Finitely Presented Groups in Geometry and Topology

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Abstract: In this paper, my main focus has been to prove that infinite \( \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2) \) is coarsely equivalent to infinite \( \bigoplus_{i=1}^{\infty} (\mathbb{Z}_3) \). Even though I haven’t been able to find an explicit proof, I’ve been able to deduct some nice properties of coarse equivalence.

1 Introduction

In topology, we usually study the properties of geometric forms that remain invariant under certain transformations, as bending or stretching. So we say that two topological spaces \( X \) and \( Y \) are homeomorphic if there exists a function between those spaces that is continuous, one-to-one, and onto, and the inverse of which is continuous. Because of continuity, it’s clear that one restrict itself to small scales. But what can we say about two countable groups?

2 Basic Definitions

Definition 2.1 A metric space consist of a pair \((X,d)\), where \(X\) is a set and \(d : X \times X \to \mathbb{R}\) is a function, called the metric or distance such that: \(\forall x, y, z \in X\)

1. \(d(x, y) \geq 0, \text{ and } d(x, y) = 0 \iff x = y : (PositiveDefiniteness)\)
2. \(d(x, y) = d(y, x) : (Symmetry)\)
3. \(d(x, z) \leq d(x, y) + d(y, z) : (TriangleInequality)\)

Definition 2.2 A metric \(d\) on a vector space \(X\) is said to be left invariant if \(\forall x, y, z \in X\)
\(d(z + x, z + y) = d(x, y)\). On a group \((G,\ast)\) we have \(d(g,h) = d(e, g^{-1} \ast h)\) \(\forall h, g \in G\)

Definition 2.3 A metric \(d\) on a metric space \(X\) is said to be proper if \(|B(x,r)| < \infty \forall x, \in X, \forall r \geq 0\)
On a group \((G,\ast)\) we have \(|B(g,r)| < \infty \forall g \in G\)
Lemma 2.4 Any proper left invariant metric on \( \mathbb{Z} \) comes from applying a function to the usual Euclidean metric of \( \mathbb{Z} \).

Proof Let \( d \) be an arbitrary proper left invariant on \( Z \), then \( d(x, y) = d(0, y - x) \) and \( \exists f : \mathbb{Z} \to \mathbb{R} \) such that \( d(0, s) = f(s) \). But \( f(-s) = d(0, -s) = d(s, 0) = d(0, s) = f(s) \) So \( d(0, y - x) = f(|y - x|) \).

Lemma 2.5 If \( d \) is a proper left invariant metric, Then \( f \) is increasing, \( f(0) = 0 \) and \( f \) is concave \( \Rightarrow f(d) \) is a proper left invariant metric.

Proof For, we just need to check the triangle inequality: 
\[
\forall x, y \in \mathbb{Z} \quad (d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z))
\]

Lemma 2.6 For any set \( A \) and a metric space \((X, d)\), we can define a metric on \( A \) by defining a injective function \( f : A \to X \) by \( d'(x, y) = d(f(x), f(y)) \) where \( x, y \in A \). \( d' \) is proper left invariant \( \iff \) \( f \) is linear.

Proof Since \( f \) is injective then \( x \neq y \) and \( d'(x, y) = 0 \) is remote. we just need to check the triangle inequality again: 
\[
d'(x, z) = d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) = d'(x, y) + d'(y, z)
\]
If \( f \) linear \( \iff f(x-y) = f(x)-f(y) \iff d'(x, y) = d(f(x), f(y)) = d(0, f(y)-f(x)) = d(0, y-x) = d'(0, y-x)
\]

Definition 2.7 A metric space \((X, d)\) is called ultrametric if
\[
\forall x, y, z \in X \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}
\]

Properties of ultrametrics \( \forall x, y, x, \in X, r, s \in R \)

- Every triangle is isosceles; i.e. \( d(x, y) = d(y, z) \) or \( d(x, z) = d(y, z) \) or \( d(x, y) = d(z, x) \).
- Every point inside a ball is its center; i.e. if \( d(x, y) < r \) then \( B(x; r) = B(y, r) \).
- Intersecting balls are contained in each other; i.e. if \( B(x; r) \cap B(y; s) \) is non-empty then either \( B(x, r) \subseteq B(y; s) \) or \( B(y, s) \subseteq B(x, r) \).

Definition 2.8 A map \( r : X \to X \) is called a retraction if \( r(x) = x, \forall x \in r(X) \). A subspace \( A \subseteq X \) is called a retraction of \( X \) if there exists a retraction on \( X \) onto \( A \).

Example \( \mathbb{Z} \hookrightarrow \mathbb{R}, z \mapsto z \) is a retraction.

Definition 2.9 A map \( f : X \to Y \) of metric spaces is called lipschitz if there is a constant \( \lambda > 0 \) such that the inequality \( d_Y(f(x), f(y)) \leq \lambda d_X(x, y) \) holds \( \forall x, y \in X \). \( f \) is called \( \lambda \) – lipschitz if we need to specify the constant \( \lambda \). \( f \) is called \( \lambda – bi \) – lipschitz if both \( f \) and \( f^{-1} \) are \( \lambda \) – lipschitz.
3 Growth of a group

Definition 3.1 Let $X$ be a group $f: \mathbb{R}(or \mathbb{Z}) \rightarrow \mathbb{Z}, r \mapsto |B_d(0,r)|$ is called the growth function of $G$. $d$ is assumed to be a proper left invariant.

Lemma 3.2 $\mathbb{Z}$ has linear growth.

Proof It’s sufficient to look at the Euclidian metric, the other metrics just depend upon it. $x \in B(0,r) \Rightarrow d(0,x) = |x - 0| = |x| \leq r$. So we have $-r, -r + 1, ..., 0, ..., r - 1, r$. Remark that if $r$ is not an integer then will be looking at the ball of radius $[r]$ and in this case we’ll have at least $(2r - 1)$ elements: still linear.

For any proper left invariant metric $\rho$ on $\mathbb{Z}$, $d'(0,x) \leq d'(0,1) + ... + d'(x-1,x) = |x|d'(0,1) = k|x|

Definition 3.3 Let $X$ be a group. $X$ is said to have quadratic growth if for any left invariant metric $d$ on $X$, $|B(0,r)| \geq q(r)$ where $q$ is a quadratic function.

Lemma 3.4 $\mathbb{Z} \oplus \mathbb{Z}$ has quadratic growth.

Proof Let $d'$ be a left invariant metric on $\mathbb{Z} \oplus \mathbb{Z}$ then
\[
d'((0,0),(x,y)) \leq d'(0,0) + ... + d'(y-1,0)
\]
\[
|u|d'(0,1) + |v|d'(0,1) \leq (|x| + |y|) \max\{u,v\}
\]
where $u = d'((0,0),(0,1))$ and $v = d'((0,0),(1,0))$
So calling $d(x,y) = |x| + |y|$(taxicab metric), we have $|B_d(0,r/\max\{u,v\}| \leq |B_{d'}(0,r)|$ and since $|B_d(0,r/\max\{u,v\}|$ has quadratic growth, that completes the proof.

Theorem 3.5 $\mathbb{Z} \oplus \mathbb{Z} \ldots \oplus \mathbb{Z}$, has order $n^{th}$ order growth.

Proof Same as the the previous 2 above.

Conjecture: $\bigoplus_{i=1}^{\infty}(\mathbb{Z})_i$ has exponential growth. Incomplete proof:
\[
\bigoplus_{i=1}^{\infty}(\mathbb{Z})_i=\bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus \bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus \bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus \ldots
\]
For any $n$, which would mean that it grows faster than any polynomial.

4 Coarse equivalence

Definition 4.1 We call a map $f: X \rightarrow Y$ of metric spaces bornologous (or large scale uniform) if there is a function $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $d_Y(f(x), f(y)) \leq \rho(d_X(x,y))$, $\forall x, y \in X$. Then $f$ is said to be $\rho$ – bornologous

Remark $\rho$ – bornologous functions are just generalization of lipschitz functions since for the lipschitz case $\rho(t) = \lambda t$
Example The application $f : \mathbb{Z} \rightarrow \mathbb{Z}, x \mapsto ax + b$ is $\rho$-bornologous where $\rho(t) = kt, \forall k \geq |a|$ since $d(f(x), f(y)) = d(ax + b, ay + b) = |a|d(x, y) \leq kd(x, y)$

Remark that function of the type $f(x) = x^\alpha$ where $\alpha \leq 1$ are $\rho$-bornologous

Definition 4.2 Two maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are said to be $C$-close if there exist $C > 0$ such that $d_Y(f(x), g(x)) \leq C, \forall x \in X$

Definition 4.3 Two metric spaces $X$ and $Y$ are said to be coarsely equivalent if there exist $f : X \rightarrow Y$ and $g : Y \rightarrow X$ that are $\rho$-bornologous and there exist $C > 0$ such that $d_X(x, gof(x)) \leq C$ and $d_Y(y, fogy) \leq C$ are said to be $C$-close if there exist $C > 0$ such that $d(f(x), g(x)) \leq C, \forall x \in X$

Example $\mathbb{Z}$ is coarse equivalent to $\mathbb{R}$
For, let $f : \mathbb{Z} \hookrightarrow \mathbb{R}, n \mapsto n$ and $g : \mathbb{R} \rightarrow \mathbb{Z}, x \mapsto \text{int}(x)$
these two functions are clearly bornologous and $d_X(n, gof(n)) = d_X(n, n) = 0 \leq 1$
$d_Y(x, \text{int}(x)) \leq 1$

An intuitive understanding of coarse equivalence is to see if two spaces that look different from a small scale, look the same if we very far away.

Proposition 4.4 Coarse equivalence is an equivalence relation.

Proof Reflexivity. If $(X, d)$ is a metric space Then define $f : X \rightarrow X, x \mapsto x$ and $g = f$ for all $x \in X$.

$\rho$-bornologous

$d(f(x), f(x')) = d(x, x')$ and same for $d(g(x), g(x')) = d(x, x')$. Setting $\rho(t) = t$,
f is $\rho$-bornologous

It is clear that $f \circ f = f \circ g = g \circ f = \text{id}_X$ are $C$-close to the identity $\forall C \geq 0$.

Symmetry. $X \sim_{\text{coarse}} Y \Rightarrow \exists f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f, g$ are $\rho$-bornologous and $\exists C > 0$ where $f \circ g, g \circ f$ are $c$-close to the identity map. Just switching the role of $f$ and $g$ gives $Y \sim_{\text{coarse}} X$.

Transitivity. Suppose $X \sim_{\text{coarse}} Y$ and $Y \sim_{\text{coarse}} Z$ Then $\exists f_1 : X \rightarrow Y, g_1 : Y \rightarrow X, f_2 : Y \rightarrow Z, g_2 : Z \rightarrow Y$ where $f_1, g_1, f_2, g_2$ are $\rho$-bornologous for some $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\exists C > 0$ such that $d(x, g_1 \circ f_1(x)) < C, d(y, f_1 \circ g_1(y)) < C$ and $d(y, g_2 \circ f_2(y)) < C$ and $d(z, f_2 \circ g_2(z)) < C$.

We need to find two functions $f : X \rightarrow Z$ and $g : Z \rightarrow X$ that are $\rho$-bornologous and such that $d_X(x, gof(x)) \leq C'$ and $d_Z(x, fogy) \leq C'$ for some $C'$. Let $f = f_2 \circ f_1$ and $g = g_1 \circ g_2$

$\rho$-bornologous

$d(f(x), f(x')) = d(f_2 \circ f_1(x), f_2 \circ f_1(x')) = d(f_2(f_1(x)), f_2(f_1(x'))) \leq \rho(d(f_1(x), f_1(x'))) \leq \rho(\rho(d(x, x'))).$ since $\rho$ is increasing.
Similarly, \( d(g(z), g(z')) = d(g_1 \circ g_2(z), g_1 \circ g_2(z')) = d(g_1(g_2(z)), g_1(g_2(z')) \leq \rho(d(g_2(z), g_2(z')) \leq \rho(\rho(d(x, x'))) = \rho \circ \rho(d(x, x')). \) since \( \rho \) is increasing.

**C - closeness**

\[ d(x, g \circ f(x)) = d(x, g(f(x))) = d(x, g_2 \circ g_1 \circ f_2 \circ f_1(x)) \leq d(x, g_1 \circ f_1(x)) + d(g_1 \circ f_1(x), g_2 \circ g_1 \circ f_2 \circ f_1(x)) ( \text{By the triangle inequality, } ) \]

And since \( Y \approx Z \) then \( d(y, g_2 \circ f_2(y)) \leq c, \) in particular for \( y = f_1(x) \). So \( d(f_1(x), g_2 \circ f_2 \circ f_1(x)) \leq c, \forall x \in X. \) Since \( g_1 \) is \( \rho - \text{bornologous,} \) applying it to the left side of the previous inequality we have \( d(g_1(g_1(f_1(x))), g_1(g_2 \circ f_2 \circ f_1(x))) \leq \rho(d(f_1(x), g_2 \circ f_2 \circ f_1(x))) \leq \rho(c). \)

So \( d(x, g_1 \circ g_2 \circ f_2 \circ f_1(x)) \leq d(x, g_1 \circ f_1(x)) + d(g_1 \circ f_1(x), g_2 \circ g_1 \circ f_2 \circ f_1(x)) \leq \rho(c). \)

Let \( C' = \rho(C) + C \) then \( d(x, g \circ f(x)) \leq C' \)

Similarly we get that \( d(z, g \circ f(z)) \leq C' \)

**Proposition 4.5** Let \( X \) and \( Y \) be two metric spaces. Then \( X \approx_{\text{coarse}} Y \Rightarrow \) the inverse image of each bounded under \( f \) set in \( Y \) is bounded. (Propersness)

**Proof** Suppose \( X \approx_{\text{coarse}} Y \) then there exist \( f : X \to Y \) and \( g : Y \to X \) such that \( f \) and \( g \) are \( \rho - \text{bornologous} \) and \( d_X(x, g(f(x))) \leq C \) and \( d_Y(y, f(g(y))) \leq C \)

Let \( S = \bigcup \{ y_i \} \) be an arbitrary closed set of \( Y \) \( f^{-1}(\{ y_i \}) = \{ x \in X | f(x) = y_i \}. \)

If \( f^{-1}(\{ y_i \}) \) contains only one element, then nothing to prove.

So suppose \( f^{-1}(\{ y_i \}) \) contains at least two elements \( x \) and \( x'. \)

If \( f^{-1}(S) \) is not bounded, then \( \forall k > 0, d(x, x') > k \) but \( d(x, x') \leq d(x, g \circ f(x)) + d(x', g \circ f(x')) \) since \( g \circ f(x) = g(f(x)) = g(f(x')) = g \circ f(x') \)

So \( d(x, x') \leq d(x, g \circ f(x)) + d(x', g \circ f(x')) \leq C + C = 2C \) Contradiction since \( d(x, x') \) supposed to be greater than any \( k > 0. \)

**Definition 4.6** Two Metric spaces \( X \) and \( Y \) are said to be bijectively coarse equivalent if there exist \( f : X \to Y \) and \( g : Y \to X \) that are bijective, \( \rho - \text{bornologous} \) and there exist \( C > 0 \) such that \( d_X(x, g(f(x))) \leq C \) and \( d_Y(y, f(g(y))) \leq C \) are said to be \( C \)-close if there exist \( C > 0 \) such that \( d(f(x), g(x)) \leq C, \forall x \in X \)

5 The \( d_L \) Metric

Let \( \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2) \), so elements in \( X \) are under the form \((a_1a_2, ..., a_n, 0, 0, 0, ....)\) where \( a_i = 0, 1 \) which means that there is some \( n \) after which the \( a_i's \) are zero. We can then compare two elements in \( X \) by looking at the largest index at which they different, this index will be said to the distance between these two points in \( X \).

**Example** let \( a = (011101000...000...) \) and \( b = (10010001000....000...) \)
then \( d(a, b) = 8 \)

**Definition 5.1** We define \( d_L(a, b) = \max \{i | a_i \neq b_i\}. \)

Let \( G \) be a loacally finite group, there is way of defining a proper left invariant metric on \( G \) by considering a filtration \( L \) of \( G \). So if consider a filtration \( L \) of \( G \)
by subgroups \( L = \{ 1 \subset G_1 \subset G_2 \subset G_3 \subset \ldots \} \), we define the metric \( d_L \) associated with this filtration as: 
\[
    d_L(x, y) = \min \{ i | x^{-1} y \in G_i \}.
\]

**Proposition 5.2** \( d_L \) is an ultrametric

**Proof** Let’s use the fact that any triangle in an ultrametric space is isosceles. Let consider \( a = (a_1, a_2, \ldots, a_n, 0, \ldots) \), \( b = (b_1, b_2, \ldots, b_m, 0, \ldots) \), \( c = (c_1, c_2, \ldots, c_l, 0, \ldots) \). We need to show that if \( d_L(a, b) \geq d_L(b, c) \) then \( d_L(a, b) = d_L(a, c) \) or \( d_L(a, b) = d_L(b, c) \).

So \( d_L(a, b) = \max \{ n, m \} \) and \( d_L(b, c) = \max \{ m, l \} \) and then \( \max \{ n, m \} \geq \max \{ m, l \} \).

We decompose this inequality in two cases:

- **Case 1:** If \( \max \{ n, m \} = n \) Then \( n \geq l \Rightarrow d_L(a, c) = n = d_L(a, b) \).
- **Case 2:** If \( \max \{ n, m \} = m \) Then \( \max \{ n, m \} = \max \{ m, l \} \Rightarrow m \geq l \) otherwise we will have \( \max \{ m, l \} = l \) and contradiction from \( \max \{ n, m \} = m \geq \max \{ m, l \} = l \).

So \( \max \{ m, l \} = m = \max \{ n, l \} \) and then \( d_L(a, b) = L(b, c) \).

**Example** A filtration \( L \) of \( \bigoplus_{i=1}^{\infty}(Z_2) \) is \( L = Z_2 \subset Z_2 \oplus Z_2 \subset Z_2 \oplus Z_2 \oplus Z_2 \subset \ldots \).

Let’s find the Growth function of \( \bigoplus_{i=1}^{\infty}(Z_2) \): \( B(0, r) = \{ a \in \bigoplus_{i=1}^{\infty}(Z_2) \} \) \( d_L(0, a) \leq r \). \( d_L(0, (a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)) \leq r \) \( \Rightarrow n \leq r \) \( \Rightarrow |B(0, r)| = 2^r \) since each time there are 2 ways of picking up the \( a_i \)’s.

Let’s find the Growth function of \( \bigoplus_{i=1}^{\infty}(Z_3) \): \( B(0, r) = \{ a \in \bigoplus_{i=1}^{\infty}(Z_3) \} \) \( d_L(0, a) \leq r \). \( d_L(0, (a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)) \leq r \) \( \Rightarrow n \leq r \) \( \Rightarrow |B(0, r)| = 3^r \) since each time there are 3 ways of picking up the \( a_i \)’s.

**Proposition 5.3** If \( (X, d) \) is a group with \( d \) the taxicab metric and \( d' \) is any proper left invariant metric other than \( d \) then \( X, d \) \( \simeq \text{coarse} \) \( (X, d') \).

**Proof** Define \( f : (X, d) \rightarrow (X, d'), x \mapsto x \) and \( g : (X, d') \rightarrow (X, d), x \mapsto x \).

\( \rho - \text{bornologous} \)

Let \( \lambda_1 = \max \{ d(0, e_i) : e_i \text{, generators } \in X \} \) and \( \lambda_2 = \frac{1}{\lambda_1} \).

Then from the proof of **Lemma 3.4**

\( d'(f(x), f(y)) = d(x, y) = d(0, y - x) \leq \lambda_1.d(0, y - x) = \lambda_1.d(x, y) \)

Similarly,

\( d(g(x), g(y)) = d(x, y) = d(0, y - x) \leq \lambda_2.d'(0, y - x) = \lambda_2.d'(x, y) \)

So \( f \) is \( \rho_1 - \text{bornologous} \) and \( g \) is \( \rho_2 - \text{bornologous} \) where \( \rho_1(t) = \lambda_1.t \) and \( \rho_2(t) = \lambda_2.t \).

\( C - \text{closeness} \)

\( d(x, g \circ f(x)) = d(x, g(x)) = d(x, x) = 0 \) and \( d'(x, f \circ g(x)) = d'(x, f(x)) = d'(x, x) = 0 \)

So it’s sufficient to take \( C = 0 \).

**Corollary 5.4** If \( G \) is a group and \( L = G_0 \subset G_1 \subset \ldots \) and \( L' = G'_0 \subset G'_1 \subset \ldots \) two filtrations of \( G \), then \( (G, d_L) \) and \( (G, d_{L'}) \) are bijectively coarse equivalent.
Proof \((G, d_L) \simeq^{\text{coarse}} (G, d)\) and \((G, d) \simeq^{\text{coarse}} (X, d_L')\)

Now since \(\simeq^{\text{coarse}}\) is an equivalence relation (proposition 4.4), we just proved that \((G, d_L) \simeq^{\text{coarse}} (G, d_L')\). It’s bijective since we’re just using the identity map on \(G\).

6 Problems and Properties

6.1 \(\bigoplus_{i=1}^{\infty} (\mathbb{Z}_2)_i\) and \(\bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i\) coarsely?

Proof Define
\[ f : \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2)_i \to \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2)_i \quad a = (a_1 a_2, ..., a_n, 0, 0, 0, ...) \mapsto (0, a_2, ..., a_n, 0, 0, 0, ...) \]
\[ g : \bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i \hookrightarrow \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2)_i \quad \text{the canonical inclusion.} \]
\(\rho\) - bornologous
\[ d(f(a), f(b)) = d((0, a_2, ..., a_n, 0, 0, 0, ...), (0, b_2, ..., b_m, 0, 0, 0, ...)) \leq \max\{m, n\} \]
And \(d(a, b) = d((a_1 a_2, ..., a_n, 0, 0, 0, ...), (b_1 b_2, ..., b_m, 0, 0, 0, ...)) = \max\{n, m\} \)
So taking \(\rho(t) = t \Rightarrow d(f(a), f(b)) \leq \max\{m, n\} \leq \rho(d(a, b)) \)
\[ d(g(a), g(b)) = d((0, a_2, ..., a_n, 0, 0, 0, ...), (0, b_2, ..., b_m, 0, 0, 0, ...)) = d(a, b) \]
So by taking \(\rho(t) = t\) we have \(d(g(a), g(b)) \leq \rho(d(a, b))\). - closeness
\[ d(a, g \circ f(a)) = d((a_1 a_2, ..., a_n, 0, 0, 0, ...), (a_2, ..., a_n, 0, 0, 0, ...)) = 1 \]
\[ d(a', f \circ g(a')) = d((0, a'_2, ..., a'_n, 0, 0, 0, ...), (0, a_2, ..., a_n, 0, 0, 0, ...)) = 0 \]
So by taking \(C = 1\) we have \(g \circ f \simeq_{\text{closeness}} 1d_{\bigoplus_{i=1}^{\infty} (\mathbb{Z}_2)_i}\) and \(f \circ g \simeq_{\text{closeness}} 1d_{\bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i}\).

6.2 If \(H_1, H_2\) are two finite groups and \(G\) a group. Then \(H_1 \oplus G \simeq^{\text{coarse}} H_2 \oplus G\)

Proof Let \(f : H_1 \oplus G \to H_2 \oplus G, (h_1, g) \mapsto (0, g)\) and
\[ g : H_2 \oplus G \to H_1 \oplus G, (h_2, g) \mapsto (0, g) \quad \text{where} \quad h_1 \in H_1, h_2 \in H_2, g \in G \]
\(\rho\) - bornologous
\[ d(f(h_1, g), (h_1', g)) = d((0, g), (0, g)) = 0 \leq d((h_1, g), (h_1', g)), \forall (h_1, g), (h_1', g) \in H_1 \oplus G \] and similarly
\[ d(f(h_2, g), (h_2', g)) = d((0, g), (0, g)) = 0 \leq d((h_2, g), (h_2', g)), \forall (h_2, g), (h_2', g) \in H_2 \oplus G \]
So taking \(\rho(t) = t\) is more than sufficient.
\(C\) - closeness
Let’s denote \(D_1 = \max\{d(h_1, h_1') : h_1, h_1' \in H_1\}\) and \(D_2 = \max\{d(h_2, h_2') : h_2, h_2' \in H_2\}\) which we know exist since and \(H_1, H_2\) are two finite groups.
Let \(C = \max\{D_1, D_2\}\) then
\[ d(h_1, g \circ f(h_1)) \leq C \quad \text{since} \quad h_1, g \circ f(h_1) \in H_1. \]
Similarly \(d(h_2, f \circ g(h_2)) \leq C \quad \text{since} \quad h_2, f \circ g(h_2) \in H_2. \]

So, \(H_1 \oplus G \simeq^{\text{coarse}} H_2 \oplus G\)

Example \(Z_a \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z}_c)_i \simeq^{\text{coarse}} Z_b \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z}_c)_i\)

Proof \(Z_a = \{\bar{0}, \bar{1}, ..., a - \bar{1}\}\) and \(Z_a = \{\bar{0}, \bar{1}, ..., b - \bar{1}\}\) The rest is the same as the previous proof by letting \(\bigoplus_{i=1}^{\infty} (\mathbb{Z}_c)_i = G\).
Example: $\bigoplus_{i=1}^{n}(Z_a)_i \oplus G \simeq^{coarse} \bigoplus_{i=1}^{m}(Z_b)_i \oplus G$

Proof: Same proof by letting $H_1 = \bigoplus_{i=1}^{n}(Z_a)_i$, and $H_2 = \bigoplus_{i=1}^{m}(Z_b)_i$.

Example: $\mathbb{Z}_3 \oplus \bigoplus_{i=1}^{\infty}(Z_2)_i \simeq^{coarse} \bigoplus_{i=1}^{\infty}(Z_2)_i$

Proof: Same by letting $H_1 = \bigoplus_{i=1}^{n}(\mathbb{Z}_3)_i$, and $H_2 = \{0\}$ and $G = \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$.

6.3 [Dr. Conant: Property $\star$]: A space $X$ is said to have property $\star$ if $\forall c > 0, \exists R > 0, \exists x_0 \in X$ such that $X \backslash B_R(x_0)$ is not c-connected.

Proposition 6.1 (Dr. Conant): If $X$ has property $\star \Rightarrow X$ is unbounded.

Proof: Suppose $X$ is not unbounded, then $D = diam(X) < \infty$, so $X \backslash B_R(x_0) \subset X$ for any $B_R(x_0)$ and then $X \backslash B_R(x_0)$ is $D - connected$. Remark that I am assuming that $X \backslash B_R(x_0) \neq \emptyset$.

6.4 Theorem [Dr. Conant]: $\mathbb{Z}$ is not coarsely equivalent to $\mathbb{Z} \oplus \mathbb{Z}$

Proof (Dr.Conant) Suppose otherwise. Then we have maps $f: \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and $g: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ and $f \circ g$ are c-close to the identity and $f$ and $g$ are $\rho - bornologous$. Let $B$ be a ball of radius $R$ centered at 0 in $\mathbb{Z}$ such that $Z$ is not $(\rho(1) + c) - connected$.

Claim 1: There exists an $S$ such that $\rho(t) \geq S - c \Rightarrow t > R$.

There are two cases. If there are no $t$ such that $\rho(R) < \rho(t)$, then choose $S$ so that $S > \rho(R) + c$. Then there are no $t$ satisfying $\rho(t) \geq S - c$, so that the implication $\rho(t) \geq S - c \Rightarrow t > R$ is vacuously true. On the other hand, if there is some $t_0$ such that $\rho(t_0) > \rho(R)$, then let $S = \rho(t_0) + c$. Then if $\rho(t) \geq S - c$, that implies that $\rho(t) \geq \rho(t_0) > \rho(R)$ which implies that $t > R$ as desired since $\rho$ is an increasing function.

Let $B' \subset Z \oplus Z$ be a ball around $f(0)$ of radius $S$, as in the previous claim.

Claim 2: We have $g(\mathbb{Z} \oplus \mathbb{Z} \backslash B') \subset \mathbb{Z} \backslash B$. Let $y \in \mathbb{Z} \oplus \mathbb{Z} \backslash B'$, then $d(y; f(0)) > S$.

Now $\rho(d(g(y); 0)) \geq d(f(g(y); f(0)) \geq |d(f(g(y); y) - d(y; f(0))| \geq S - c$

So by claim 1, with $t = d(g(y); 0)$, we have $d(g(y); 0) > R$. So $g(y) \in Z \backslash B$.

Now choose $x, x' \in Z$ which cannot be connected by a $c + \rho(1) - chain$, and which are sufficiently far from $\{0\}$ such that $f(x); f(x') \notin B'$. (Since $B'$ is bounded, we know that $f^{-1}(B)$ is a bounded set.) In $g(\mathbb{Z} \oplus \mathbb{Z} \backslash B')$, we can connect $f(x); f(x')$ by a 1-chain:

$f(x) = y_0; y_1; y_2; \ldots; y' = f(x')$ where $d(y_i; y_{i+1}) \leq 1$. Then we know $g(f(x)); g(y_0); \ldots; g(y_{i-1}); g(f(x_0))$ is a $\rho(1) - chain$, which lies in $\mathbb{Z} \backslash B$.

Prepending $x$ to the beginning and appending $x'$ to the end, we get a $c + \rho(1)$ chain connecting $x$ to $x'$, which is a contradiction.
6.5 Question [Dr. Conant]: Is the growth function conserved by Coarse equivalence? i.e. $X \simeq_{\text{coarse}} Y$ and $X$ has $n^{th}$ order growth degree $\Rightarrow Y$ has $n^{th}$ order growth.

It works for $Z$ and $Z \oplus Z$ since $Z$ has linear growth and $Z \oplus Z$ has quadratic growth and that they are not coarsely equivalent. But about in general?

6.6 Question [Dr. Conant]: Is Property $\star$ conserved by Coarse equivalence? i.e. $X \simeq_{\text{coarse}} Y$ and $X$ has Property $\star$ $\Rightarrow Y$ has Property $\star$.

It works for $Z$ and $Z \oplus Z$ since $Z$ has Property $\star$ and $Z \oplus Z$ doesn’t have Property $\star$ and that they are not coarsely equivalent. But about in general?

References
