Many economic concepts can be expressed as functions (e.g., cost, utility, profit),

- To be optimized (maximized or minimized),
- Subject to or not subject to constraints (e.g., income or resource constraints).

Optimizing a function of one variable (unconstrained):
Maximization (one variable and no constraint)

- To **maximize** \( \pi = f(q) \), look at changes in \( \pi \) as \( q \) increases.
- As long as \( \Delta \pi > 0 \), \( \pi \) increases.
- Can express \( \Delta \)'s as a ratio \( \frac{\Delta \pi}{\Delta q} \); could be + or -.
- Limit of ratio as \( \Delta q \to 0 \) is derivative of \( \pi = f(q) \).
- For \( q < q^* \), derivative > 0.
- For \( q > q^* \), derivative < 0.
- At \( q^* \), derivative = 0.
Minimization (one variable and no constraint)

- To **minimize** $AC = f(q)$, look at $\Delta AC$ as $q$ increases.
- As long as $\Delta AC < 0$, $AC$ is decreasing. The change ratio is
  \[
  \frac{\Delta AC}{\Delta q}
  \]
  Limit of ratio as $\Delta q \to 0$ is derivative of $AC = f(q)$.
  For $q < q^{**}$, derivative $< 0$.
  For $q > q^{**}$, derivative $> 0$.
  At $q = q^{**}$, derivative $= 0$.
  \[
  AC = f(q)\text{ is convex.}
  \]
  \[
  \frac{dAC}{dq} < 0 \quad \frac{dAC}{dq} > 0
  \]
  $AC = f(q)$
• First derivative is the slope of a line tangent to the function at a point.
• Derivative is denoted as:
  \[ \frac{d\Pi}{dq} \text{ or } \frac{dAC}{dq} \text{ or } \frac{df}{dq} \text{ or } f'(q) = f' \]
• To maximize or minimize \( f(q) \), find \( q \) value where:
  \[ \frac{df}{dq} = 0 \]
• Derivative will be a constant if the original function is linear; thus, the function has no Max or Min.
• But if the function is nonlinear, its derivative will be a function of \( q \) itself (the derivative varies as \( q \) varies).
• For derivative at a specific value of \( q \), say \( q_1 \), denote
  \[ \frac{d\Pi}{dq} \bigg|_{q=q_1} \cdot \frac{d\Pi}{dq} \bigg|_{q=q^*} = 0. \]
To find the \( q \) where \( \Pi \) is maximized or AC is minimized, find the derivative of \( f(q) \), and set it equal to 0 and solve for \( q \) to get \( q^* \).

The “**First Order Condition (FOC)**” for a maximum or a minimum of an unconstrained function of one variable is:

\[
\text{FOC is } \frac{df}{dq} = 0. \quad \text{Solve the FOC for } q \text{ to get } q^* \text{ (the optimal } q). 
\]

The FOC is a necessary condition for a maximum (or minimum) but not a sufficient condition.
Second Order Condition (SOC)

The FOC is $dy/dx = 0$.

FOC is necessary but not sufficient

Take the derivative (second derivative) of the derivative (first derivative).
Examples:

\[ \frac{dy}{dx} = 0 \]

\[ \frac{d^2y}{dx^2} = 0 \]

Local maximum and local minimum

\[ \frac{dy}{dx} > 0 \]

\[ \frac{d^2y}{dx^2} < 0 \]

\[ \frac{dy}{dx} < 0 \]

\[ \frac{d^2y}{dx^2} > 0 \]

\[ \frac{dy}{dx} > 0 \]

\[ \frac{d^2y}{dx^2} < 0 \]
An inflection point is where the second derivative equals zero when evaluated at the $q^*$ identified by the FOC.

When the first derivative equals zero, you have a minimum, maximum, or an inflection point. Not all inflections have $f' = 0$.

\begin{align*}
\text{SOC} & \quad \text{if } f'' \bigg|_{q^*} < 0, \text{ have a maximum at } q^* \\
& \quad \text{if } f'' \bigg|_{q^*} > 0, \text{ have a minimum at } q^* \\
& \quad \text{if } f'' \bigg|_{q^*} = 0, \text{ have an inflection at } q^* \\
\end{align*}

For an unconstrained function of one variable:

\begin{align*}
\text{Max} & \quad f' = 0 \\
\text{Min} & \quad f' = 0 \\
\text{SOC} & \quad f'' \bigg|_{q^*} < 0 \\
& \quad f'' \bigg|_{q^*} > 0 \quad \text{Solve FOC for } q^* \text{ and evaluate } f'' \text{ at } q^*.
\end{align*}
Rules for Finding Derivatives

1. The derivative of a constant is 0. If \( b \) is a constant in \( y = b \), then \( y = a + bx \), the derivative of \( a \) with respect to \( x \) is 0.

\[
\frac{dy}{dx} = \frac{db}{dx} = 0.
\]

2. The Power Rule – The derivative of a variable raise to a power is

\[
\frac{d(ax^b)}{dx} = bax^{b-1}, \quad \frac{d(ax)}{dx} = ax^{1-1} = a,
\]

\[
\frac{dx}{dx} = 1, \quad \frac{dx^b}{dx} = bx^{b-1}.
\]
• Examples:

For $y = a+bx^2$

\[\frac{dy}{dx} = 2bx\]
\[\frac{d^2y}{dx^2} = 2b\]

$\Rightarrow x^* = 0$

$f''|_{x^*} = 2b$

$y = a+bx+cx^2+ex^3$

\[\frac{dy}{dx} = b + 2cx + 3ex^2\]

$\Rightarrow x^* = 0$

Anything raised to the 0 power is 1!

\[\frac{d^2y}{dx^2} = 2c + 6ex\]

Review of the FOC and SOC for an unconstrained function of one variable:

Find $x^*$ from the FOC ($\frac{dy}{dx} = 0$), then look at SOC by substituting $x^*$ into $\frac{d^2y}{dx^2}$ to see if second derivative evaluated at $x^*$ is $>$, $<$, or $= 0$. 
The derivative of the natural log of $x$ is the inverse of $X$, \[
\frac{dy}{dx} = \frac{d \ln x}{dx} = \frac{1}{x}.
\]

The actual rule is \(\frac{d \ln(x^c)}{dx} = \frac{bc}{x}\) because \[
\frac{d \ln x}{dx} = \frac{c}{x}
\]

\(\ln(x^c) = c \ln x\) and \(\frac{d c \ln x}{dx} = \frac{c}{x}\)

\(b \ln(x^c) = b c \ln x\) and \(\frac{d b c \ln x}{dx} = \frac{bc}{x}\)

If \(y = 3 + 5 \ln(x^2)\), \(\frac{dy}{dx} = \frac{5 \cdot 2}{x}\), May not always have the parentheses.
The exponential function rule -
For \( y = a^{bx} \), \( \frac{dy}{dx} \) = the derivative of the exponent (bx) times the original function \( a^{bx} \) times the natural log of the constant (a).

\[
\frac{da^{bx}}{dx} = \frac{dbx}{dx} a^{bx} \ln a = ba^{bx} \ln a, \quad \frac{da^x}{dx} = a^x \ln a
\]

Can give constants different labels than above.
If \( y = ac^x \), then \( \frac{dy}{dx} = ac^x \ln c \).

\[
\frac{de^{bx}}{dx} = be^{bx} \ln e = be^{bx} (1) = be^{bx}, \quad \frac{de^x}{dx} = e^x \ln e = e^x (1) = e^x
\]

e is the base of natural logarithms = 2.71828, so that \( \ln e = 1 \).
The Sum of Two Functions Rule –

$$\frac{d[f(x) + g(x)]}{dx} = f'(x) + g'(x).$$

The derivative of the sum of two functions of $x$ is the sum of derivatives of the two functions:

Example: If $y = ax + bx^2$, then $\frac{dy}{dx} = a + 2bx$. 
The Product Rule — The derivative of the product of two functions of \( x \) is the first times the derivative of the second plus the second times the derivative of the first.

\[
\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)
\]

Example: \( y = (2x-1)(2-9X^2) \)

\[
\frac{dy}{dx} = (2x - 1)(-18X) + (2 - 9x^2)(2)
\]

\[
= (-36x^2 + 18x) + (4 - 18x^2)
\]

\[
= 4 + 18x - 54x^2
\]
The Quotient Rule - The derivative of the quotient of two functions of x is the derivative of the top times the bottom minus the top times the derivative of the bottom all divided by the bottom squared.

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

Ex 1: If \( y = \frac{x^3}{x^2 + 2} \), \( \frac{dy}{dx} = \frac{3x^2(x^2 + 2) - x^3(2x)}{x^4 + 4x^2 + 4} = \frac{x^4 + 6x^2}{x^4 + 4x^2 + 4} \)

Ex 2: If \( y = \frac{1}{x^4} \), \( \frac{dy}{dx} = \frac{0(x^4) - 1(4x^3)}{x^8} = \frac{-4x^3}{x^8} = \frac{-4}{x^5} = -4x^{-5} \)

Or \( y = \frac{1}{x^4} = x^{-4} \), \( \frac{dy}{dx} = -4x^{-5} \)
The Chain Rule - Allows differentiation of complex functions in terms of elementary functions.

If \( y = f(x) \) and \( x = g(z) \) so that \( y = f(g(z)) \), then

\[
\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{df}{dg} \cdot \frac{dg}{dz} = \frac{df}{dx} \cdot \frac{dx}{dz}
\]
Examples:

1. \( y = (z^2 + 2z)^{\frac{1}{3}} \), \( x = z^2 + 2z \), so \( y = x^{\frac{1}{3}} \)

\[
\frac{dy}{dz} = \left( \frac{dy}{dx} = \frac{1}{3} (z^2 + 2z)^{-\frac{2}{3}} \right) \left( \frac{dx}{dz} = 2z + 2 \right)
\]

\[
= \frac{1}{3} (z^2 + 2z)^{-\frac{2}{3}} (2z + 2) = \frac{2(z+1)}{3(z^2 + 2z)^{\frac{2}{3}}}
\]
Examples:

2. \( y = (3x + 1)^2, \quad \frac{dy}{dx} = 2(3x + 1)3 = 18x + 6 \)

3. \( y = \ln(x^2 - 4x + 2) \)
   \[
   \frac{dy}{dx} = \frac{1}{x^2 - 4x + 2} \cdot \left(2x - 4\right) = \frac{2x - 4}{x^2 - 4x + 2}
   \]
Functions of More than One Variable

"Partial" derivatives – May want to know how \( y \) changes as one of several \( x \) variables changes, ceteris paribus (as in demand curve).

\[
y = f(x_1, x_2, x_3, \ldots, x_n)
\]

Several derivatives can be taken; one with respect to each \( X_i \) holding the other variables constant. Denoted as:
\[
\frac{\partial y}{\partial x_i} = f_{x_i} = f_i
\]

When taking this derivative, all other \( x_i \)'s are treated as constants. Fortunately, the rules for derivatives of functions of one variable apply to partial derivatives also. We simply treat the other \( x \)'s as constants.
**Second Order Partial Derivatives** are analogous to the early one-variable case. Denoted as:

\[
\frac{\partial^2 y}{\partial x_i \partial x_j} = f_{ij} \quad \text{“cross second partial”} \quad i \neq j
\]

\[
\text{“own second partial”} \quad i = j
\]

There are four combinations in the two variable case, \(f_{ij}, f_{ii}, f_{ji}, \text{ and } f_{jj}\); but \(f_{ij}\) and \(f_{ji}\) are equal (Young’s Theorem). The order of differentiation does not matter.

**Partial derivatives reflect the units of measurement for \(y\) and \(x\).** If \(y = \) quantity of apples demanded per year in 100 pound units and \(x\) is the price of apples per pound, \(\partial y/\partial x\) measures the change in the quantity of apples demanded in 100 pound units per year for a dollar change in the apple price, *ceteris paribus*. If demand were measured in pounds per year, the partial derivative would be 100 time larger than when demand is measured in 100 pounds per year.
Example 1: \( Q = 1000 - 20P^2 + 36I^2 \)

\[
\frac{\partial Q}{\partial P} = -40P, \quad \frac{\partial^2 Q}{\partial P \partial I} = 0, \quad \frac{\partial^2 Q}{\partial P \partial P} = -40
\]

\[
\frac{\partial Q}{\partial I} = 72I, \quad \frac{\partial^2 Q}{\partial I \partial P} = 0, \quad \frac{\partial^2 Q}{\partial I \partial I} = 72
\]

Example 2: \( Q = 1000 - 20P^2 + 36I^2 + IP \)

\[
\frac{\partial Q}{\partial P} = -40P + I, \quad \frac{\partial^2 Q}{\partial P \partial I} = 1, \quad \frac{\partial^2 Q}{\partial P \partial P} = -40
\]

\[
\frac{\partial Q}{\partial I} = 72I + P, \quad \frac{\partial^2 Q}{\partial I \partial P} = 1, \quad \frac{\partial^2 Q}{\partial I \partial I} = 72
\]
Total Differential

The total differential shows what happens to $y$ as both $x_1$ and $x_2$ change simultaneously.

The total differential is

$$dy = \frac{\partial f}{\partial x_1} \, dx_1 + \frac{\partial f}{\partial x_2} \, dx_2$$

To find a maximum or minimum for $y$ (where $dy = 0$ as $x_1$ and $x_2$ change), the FOC is that all $\frac{\partial f}{\partial x_i} = 0$ because then $dy = 0$! That is, all the first partial derivatives must be 0. This happens at the “top of the Hill” in three dimensions or the “bottom of the hole”.

\[ \begin{array}{c}
\text{f}_1 \\
\text{x}_1
\end{array} \]
Example 1: To find max. or min. for \( y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5 \), we must find where both first partials equal zero (FOCs) and solve the FOCs simultaneously for \( x_1^* \) and \( x_2^* \).

FOC 1: \[ \frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0 \quad \Rightarrow \quad 2x_1 = 2 \quad \Rightarrow \quad x_1^* = 1 \]

FOC 2: \[ \frac{\partial y}{\partial x_2} = -2x_2 + 4 = 0 \quad \Rightarrow \quad 2x_2 = 4 \quad \Rightarrow \quad x_2^* = 2 \]

Plug these optimal values into \( y \) to get \( y^* = 10 \).

Example 2: Optimize \( y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5 - x_1x_2 \),

FOC 1: \[ \frac{\partial y}{\partial x_1} = -2x_1 + 2 - x_2 = 0 \quad \Rightarrow \quad x_2 = -2x_1 + 2 \]

FOC 2: \[ \frac{\partial y}{\partial x_2} = -2x_2 + 4 - x_1 = 0 \quad \Rightarrow \quad -2(-2x_1 + 2) + 4 - x_1 = 0 \]
\[ \Rightarrow x_1^* = 0 \quad \text{and} \quad x_2^* = 2 \quad \quad \text{Plug these optimal values into } y \text{ to get } y^* = 9. \]

These FOCs are not sufficient; could be a maximum, a minimum, or a saddle point. All we know is that it is a flat spot! Need SOCs (later).
Implicit Functions

• Usually express functions **explicitly** as
  \[ y = f(x_1, x_2, \ldots, x_n) \]
  where \( y \) is a function of \( x_i \).

• However, some functions will be expressed **implicitly** as
  \[ 0 = f(x_1, x_2, \ldots, x_n, y) \]
  where there is no dependent variable.

• Can find:

  \[
  \frac{dy}{dx_i} = -\frac{f_i}{f_y} \quad \text{and} \quad \frac{dx_i}{dy} = -\frac{f_y}{f_i} \quad \text{or} \quad \frac{dx_i}{dx_j} = -\frac{f_j}{f_i}
  \]

  Remember the negative sign.
Example

\[ f(x, y) = x^2 + y^2 - 2x + 4y + 4 = 0 \]

\[ f_x = 2x - 2 \quad f_y = 2y + 4 \]

\[ \frac{dy}{dx} = -\frac{2x - 2}{2y + 4} = -\frac{x - 1}{y + 2} = \frac{1 - x}{y + 2} \]

\[ \frac{dx}{dy} = -\frac{2y + 4}{2x - 2} = \frac{y + 2}{1 - x} \]

Remember the sign!

Examples of implicit functions are production possibility curves, indifference curves, budget constraints.
Envelope Theorem

• Deals with how the optimal value of the function ($y^*$) changes as the value of one of the parameters changes (What are parameters?)

• Theorem states that one can calculate by holding independent variables at their optimal values in the function and finding $\frac{dy^*}{da}$. $y^* = f(x_1^*, x_2^*, ..., x_n^*)$
Example: \( y = -x^2 + ax + bx \)

First, find the FOC: \( \frac{dy}{dx} = -2x + a + b = 0 \),

Second, solve FOC for \( x \):

\[
x^* = \frac{a + b}{2}
\]

Third, substitute \( x^* \) into \( y \) to get:

\[
y^* = -\left(\frac{a + b}{2}\right)^2 + a\left(\frac{a + b}{2}\right) + b\left(\frac{a + b}{2}\right) = \frac{(a + b)^2}{4}
\]

Forth, partially differentiate \( y^* \) with respect to \( a \):

\[
\frac{\partial y^*}{\partial a} = \frac{a + b}{2}
\]

(Partial derivative because \( b \) is constant)
Example cont’d

From previous slide \( \frac{\partial y^*}{\partial a} = \frac{a + b}{2} \)

Thus, \( \frac{\partial y^*}{\partial a} \) depends only on values of \( a \) and \( b \), but no variables!

Assume \( b=2 \) then:

\[
\frac{\partial y^*}{\partial a} = \frac{a + 2}{2} = \left\{
\begin{array}{l}
a = 4, \quad \frac{\partial y^*}{\partial a} = 3 \\
a = 2, \quad \frac{\partial y^*}{\partial a} = 2 \\
a = 0, \quad \frac{\partial y^*}{\partial a} = 1
\end{array}
\right.
\]

In many-variable case, solve all FOC simultaneously for \( x_i^*(a) \) and substitute into \( y \) to get \( y^* = f(x_1^*, x_2^*, \ldots, x_n^*) \), then \( \frac{dy}{da} = \frac{\partial y^*}{\partial a} \).
Constrained Optimization

• Because we are looking at scarce resources, we usually must find an optimum value subject to one or more constraints (limits). These may prevent us from achieving the maximum (or minimum) of the objective function.
Lagrangian Multiplier Method

Mathematical “voodoo” to short-cut optimization subject to a constraint. The following is an important format for setting up a constrained optimization problem.

Max (Min): $y = f(x_1, x_2, x_3, \ldots, x_n)$  

Objective function to be optimized

s.t. $0 = g(x_1, x_2, x_3, \ldots, x_n)$  

Constraint

Implicit form!

Set up $L = f(x_1, x_2, x_3, \ldots, x_n) + \lambda g(x_1, x_2, x_3, \ldots, x_n)$

Where $\lambda$ is the Lagrangian multiplier.

Constraint example: $x_1 + 2x_2 + x_3 = 3$

$0 = 3 - x_1 - 2x_2 - x_3 = g(x_1, x_2, x_3, \ldots, x_n)$
Find FOC for each $x_i$ and $\lambda$:

\[
\frac{\partial L}{\partial x_1} = f_1 + \lambda g_1 = 0
\]
\[
\frac{\partial L}{\partial x_2} = f_2 + \lambda g_2 = 0
\]
\[
\frac{\partial L}{\partial x_3} = f_3 + \lambda g_3 = 0
\]
\[
\frac{\partial L}{\partial x_n} = f_n + \lambda g_n = 0
\]
\[
\frac{\partial L}{\partial \lambda} = g(x_1, x_2, ..., x_n) = 0
\]

FOC
Lagrangian Multiplier Method (cont)

Solve these $n + 1$ FOCs simultaneously to find values for $x_1^*, x_2^*, x_3^*, ..., x_n^*$, and $\lambda^*$.

These are optimum values of the $x_i$s given the constraint.

Substituting the $x_i^*$ into $y$ will give $y^*$.

These conditions are necessary, but not sufficient for an optimum. Look at Second Order Conditions (SOC) later.
You can use the Lagrangian method for any number of $x_i$ (i=1...n) and for any number of constraints. Just add another $\lambda_j$ (j=1...m) for each constraint, $g_j$ (j=1...m).

$L = f(x_1,\ldots, x_n) + \lambda_1 g_1(x_1,\ldots, x_n) + \lambda_2 g_2(x_1,\ldots, x_n)$
Interpretation of $\lambda$

Solve the first FOC for $\lambda$ to get:

$$f_1 + \lambda g_1 = 0, \lambda g_1 = -f_1, \lambda = \frac{f_1}{-g_1}$$

Do it for all $n$ FOC to get: $\lambda = \frac{f_i}{-g_i}$

Because all ratios equal $\lambda$, all ratios are equal.

$$\lambda = \frac{f_1}{-g_1} = \frac{f_2}{-g_2} = \ldots = \frac{f_n}{-g_n}$$
• At optimum (Max or Min of \( y \)), the negative of the ratios of the partial derivatives (objective/constraint) with respect to each \( x_i \) are equal and equal to \( \lambda \).

• The \( f_i \) shows the marginal contribution (benefit) of \( x_i \) to \( y \) (the objective function), while the \( -g_i \) shows the marginal cost of \( x_i \) in the constraint. That is, \( -g_i \) shows how much the other \( x_i \)s must be reduced when \( x_i \) is increased slightly, and therefore, how much \( y \) is foregone if the constraint is to hold.
• Because $\lambda = \text{the ratio of marginal benefit to marginal cost for all } x_i$, all these ratios are equal. This condition is **necessary** for a maximum (or minimum). That is, the Margin Benefit/Marginal Cost ratios must be equal for all $x_i$ and equal to $\lambda$. If one $x_i$ ratio were larger than the others, then more $x_i$ should be used and less of other $x_i$s should be used until the ratios are equal again.
• \( \lambda^* \) indicates how \( y^* \) would be affected by relaxing the constraint slightly. \( \lambda^* \) is the shadow price for a unit of the resource represented by the constraint (the value of an additional unit of the resource). If \( \lambda^* = 0 \), the constraint is redundant and the optimum is unconstrained. An additional unit of the resource would be worth nothing in regard to increasing \( y \).

• If \( \lambda^* \) is large, relaxing the constraint would be very beneficial because the benefit/cost ratio is high.

• Relaxing the constraint refers to increasing (decreasing) the value of the constraint in a maximization (minimization) problem!
Duality

• Any constrained Max (Min) problem (called the **primal** problem) is associated with a constrained Min (Max) problem (called the **dual** problem).
  
  – **Primal (Dual)** may be: **Max** Utility s.t. Income; or **Min** Cost s.t. Output level.
  
  – **Dual (Primal)** would be: **Min** Expenditures s.t. Utility level; or **Max** Output s.t. Cost level.

**Always** two ways to look at any constrained optimization problem.
Example: Primal

Min: \[ f(x, y) = z = x^2 - y^2 + 3xy + 5x \]

s.t. \[ x - 2y = 0 \]

\[ L = x^2 - y^2 + 3xy + 5x + \lambda(0 - x + 2y) \]

1. \[ \frac{\partial L}{\partial x} = 2x + 3y + 5 - \lambda = 0 \]

2. \[ \frac{\partial L}{\partial y} = -2y + 3x + 2\lambda = 0 \] \quad \text{FOCs}

3. \[ \frac{\partial L}{\partial \lambda} = -x + 2y = 0 \]
Primal example (cont.) Simultaneous Solution

1. Multiply FOC 1 by 2 and add it to FOC 2.  \[ 4x + 6y + 10 - 2\lambda = 0 \]
   \[ 3x - 2y + 0 + 2\lambda = 0 \]
   \[ 7x + 4y + 10 = 0 \]

2. Solve FOC 3 for \( x \) to get \( x = 2y \).

3. Substitute \( x \) into above result to get: \( 7(2y) + 4y + 10 = 0 \).
   This gives: \( y^* = -5/9 \).

4. Substitute \( y^* \) into \( x = 2y \) from FOC 3 to get: \( x^* = 2(-5/9) = -10/9 \).

5. Substitute \( x^* \) and \( y^* \) into FOC 1 to get: \( \lambda^* = 10/9 \).

6. Substitute \( x^* \) and \( y^* \) into \( z \) to get: \( z^* = -225/81 \).

7. Optimal solution is: \( x^* = -10/9, \ y^* = -5/9, \ z^* = -225/81, \lambda^* = 10/9 \).

If you could relax the constraint by one unit (decrease it by one unit from 0 to -1), \( \lambda^* \) shows that \( z^* \) would decrease by 10/9 of a unit.
**Example: Dual**

Max: \( w = x - 2y \)  

s.t.: \( x^2 - y^2 + 3xy + 5x = -225/81 \)

The constraint in implicit form is: \( 0 = -225/81 - f(x,y) \).

\[
L = x - 2y + \lambda^D ( -225/81 - x^2 + y^2 - 3xy - 5x )
\]
Dual example (cont.)

\[
\frac{\partial L}{\partial x} = 1 - 2\lambda^D x - 3\lambda^D y - 5\lambda^D = 0
\]

\[
\frac{\partial L}{\partial y} = -2 + 2\lambda^D y - 3\lambda^D x = 0
\]

\[
\frac{\partial L}{\partial \lambda^D} = -\frac{225}{81} - x^2 + y^2 - 3xy - 5x = 0
\]

Solve these FOCs simultaneously to get:

\[x^* = -10/9, \quad y^* = -5/9, \quad \lambda^{D*} = 9/10, \quad w^* = 0\]

If we could relax (increase) the constraint by 1 unit, the optimal value of the objective function \((w^*)\) would increase by 9/10 of a unit. Notice that values for \(x^*\) and \(y^*\) are common to the primal and dual, while the value of \(\lambda^{D*}\) for the dual is the reciprocal of \(\lambda^*\) for the primal.
Envelope Theorem in Constrained Max.

- Theorem can be used as discussed earlier to examine how a function’s optimum value \( y^* \) changes as a parameter in the problem changes. In the case at hand, differentiate optimal Lagrangian expression with respect to the parameter:

\[
\begin{align*}
\text{Max: } & y = f(x_1, x_2; a) \\
\text{st: } & 0 = g(x_1, x_2; a) \\
\end{align*}
\]

\[
L = f(x_1, x_2; a) + \lambda g(x_1, x_2; a)
\]

Solve FOC for \( x_1^*, x_2^* \) and substitute into \( L \) to get \( L^* \), which is a function of parameters only.
Envelope Theorem ...(cont.)

\[
\frac{dy^*}{da} = \frac{\partial L^*}{\partial a}
\]

Only difference is that \( y^* \) is a constrained optimum.

If \( a \) is the value of the constraint, then \( \frac{dy^*}{da} = \lambda^* = \frac{\partial L^*}{\partial a} \) is the shadow price of the resource in the constraint.

See the section in the text on Inequality Constraints (pp. 45-47) for the Lagranian methods for solving problems with inequality constraints.

**Warnings:**

Not all optimization problems can be solved with calculus. Functions must be continuous at optimum. If not, use Linear Programming methods.
Further discussion of FOC and SOC

We have already talked about FOC and SOC for functions of one independent variable and no constraints: $y = f(x)$.

\[
\begin{align*}
\text{FOC} & \quad f' = 0 \\
\text{SOC} & \quad f''|_{x^* < 0} \text{concave} \\
\end{align*}
\]

\[
\begin{align*}
\text{FOC} & \quad f' = 0 \\
\text{SOC} & \quad f''|_{x^* > 0} \text{convex} \\
\end{align*}
\]
Suppose \( y = f(x_1, x_2) \)

Two independent variables, no constraint.

FOC \[
\begin{align*}
    f_1 &= 0 \\
    f_2 &= 0
\end{align*}
\]  
Indicates a flat spot—max, min, saddle point.

Logic suggests that the own second partials evaluated at \( x_1^* \) and \( x_2^* \) would indicate which (max or Min), but…..

\[
\begin{align*}
    f_{11}|_{x_1^*, x_2^*} < 0 & \quad \text{soc} \\
    f_{22}|_{x_1^*, x_2^*} < 0 & \quad \text{Max} \\
    f_{11}|_{x_1^*, x_2^*} > 0 & \quad \text{Min} \\
    f_{22}|_{x_1^*, x_2^*} > 0 & \quad \text{Assumes only } x_1 \text{ or } x_2 \text{ is changing!}
\end{align*}
\]
However, what if both $x_1$ and $x_2$ are changing? Must consider a third SOC condition (partial derivatives are evaluated at $X_1^*$ and $X_2^*$ for SOCs):

$$f_{11} f_{22} - f_{12}^2 > 0$$

This condition must hold when evaluated at the optimal values of $x_1$ and $x_2$. This condition is one of three sufficient conditions (SOCs) when $f$ is a function of two variables (unconstrained).
Summary of FOC and SOC for functions of two independent variables and no constraint.

\( y = f(x_1, x_2) \)

**FOC for maximum or minimum**
\[
\begin{align*}
    f_1 &= 0 \\
    f_2 &= 0
\end{align*}
\]
max, min, saddle point

**SOC for a maximum:**
\[
\begin{align*}
    f_{11} < 0, f_{22} < 0 \\
    f_{11} f_{22} - f_{12}^2 > 0
\end{align*}
\]
Concave function

**SOC for a minimum:**
\[
\begin{align*}
    f_{11} > 0, f_{22} > 0 \\
    f_{11} f_{22} - f_{12}^2 > 0
\end{align*}
\]
Convex function
• \( f_1 \) and \( f_2 \) locate a flat spot.

• \( f_{11} \) and \( f_{22} \) distinguish between a minimum and a maximum.

• Think of a saddle in case of a minimum. The flat spot on the saddle is not the lowest point on the saddle.

• \( f_{11}f_{22} - f_{12}^2 > 0 \) confirms that it is not a saddle point.
Constrained Optimization

The SOC for constrained maximization of a function of two variables, subject to a linear constraint (or its dual) is:

\[ \frac{\partial L}{\partial x_1} = 0 \]
\[ \frac{\partial L}{\partial x_2} = 0 \]
\[ \ldots \]
\[ \frac{\partial L}{\partial x_n} = 0 \]
\[ \frac{\partial L}{\partial \lambda_1} = 0 \]
\[ \ldots \]
\[ \frac{\partial L}{\partial \lambda_m} = 0 \]

If a constrained function has this property, it is strictly quasi-concave.

In economic theory, we usually use this problem (e.g., Max. a utility function, s.t. a linear budget constraint (primal), or Min. linear expenditures, s.t. a fixed level of the utility function (dual).

The SOC for constrained minimization of a function of two variables, subject to a linear constraint (or its dual) is:

\[ \frac{\partial L}{\partial x_1} = 0 \]
\[ \frac{\partial L}{\partial x_2} = 0 \]
\[ \ldots \]
\[ \frac{\partial L}{\partial x_n} = 0 \]
\[ \frac{\partial L}{\partial \lambda_1} = 0 \]
\[ \ldots \]
\[ \frac{\partial L}{\partial \lambda_m} = 0 \]

If a constrained function has this property, it is strictly quasi-convex.

Solve the FOC simultaneously to get the optimal solution and plug the \( x_i^* \) into the appropriate expression above (SOC) to see if it is satisfied (or just determine the signs of the partial derivatives and determine if the expression has the appropriate sign).