This triangle defines the results we want. We can write Eq. 15.14 as
\[ x(t) = E + (k_1^2 + k_2^2) e^{-\frac{5}{2}n_0t} \left[ \sin(\omega_0 t + \theta) \right] \]
or
\[ x(t) = E + (k_1^2 + k_2^2) e^{-\frac{5}{2}n_0t} \left[ \sin(\omega_0 t + \theta) \right] \quad \text{Eq. 15.18} \]

\( k_1 \) and \( k_2 \) are determined using initial conditions associated with the 2nd order differential equation.

Suppose that we say that for this case, all initial conditions are zero, \( E = 1 \).
In particular, that \( x(0) = 0 \), \( \frac{dx(0)}{dt} = 0 \).

From this we find:
\[ k_1 = -1, \quad k_2 = \frac{-1}{\sqrt{1 - \xi^2}} \quad \text{Eq. 15.19} \]

so that:
\[ \frac{k_1^2 + k_2^2}{\sqrt{1 - \xi^2}} = \frac{1}{\sqrt{1 - \xi^2}} \quad \text{Eq. 15.20} \]

Equation 15.18 becomes
\[ x(t) = 1 + \frac{e^{-\frac{5}{2}n_0t}}{\sqrt{1 - \xi^2}} \sin(\omega_0 t + \theta) \quad \text{Eq. 15.21} \]

\[ \theta = \tan^{-1}\left( \frac{k_1}{k_2} \right) = \tan^{-1}\left( \frac{\sqrt{1 - \xi^2}}{\xi} \right) \quad \text{Eq. 15.20} \]
also,
\[ \theta = \cos^{-1} \left( \xi \right) \]
For any given 2nd order differential equation we can find $y$ and $w_n$.

The general nature of the response of Eq. 15.21 is shown in Fig. 15.5. We want to know the time to the first peak of this response, if we take

$$\frac{dy(t)}{dt} = 0$$

and solve for $t_p$ we find

$$t_p = \frac{\pi}{w_n \sqrt{1 - \frac{y}{3}}}$$

Eq. 15.21

We can also find that the amount of overshoot is

$$0.5 = (100) e^{-\frac{\pi}{\sqrt{1 - \frac{y}{3}}}}$$

Eq. 15.22

Eq. 15.21 and 15.22 are extremely important in systems. They pretty much characterize the step response of a system.
The previous notes can be considered as general background for 2nd order circuits. We now turn to some examples. We start with cases in which the circuit is overdamped. Then we graduate to critical damped and finally, underdamped.

**Example 15.2**

You are given the circuit shown in 

*Figure 15.7.*

![Circuit Diagram](image)

*Figure 15.7: Circuit for Example 15.2.*

For $t < 0$

![Circuit Diagram](image)

*Figure 15.8: Circuit of Example 15.2, $t < 0$.*
We see that:
\[ i(t) = 0, \quad V(0) = 35 \quad V = V(0^+) \]

Figure 15.9: Circuit of Example 15.2, \( t > 0 \).

The above circuit can be simplified as shown in Figure 15.10.

Figure 15.10: Circuit in simplified form for Example 15.2.

We write the following equations:

\[ 15 i(t) + 3 \frac{d^2 v}{dt^2} + v(t) = 25 \quad \text{Eq. 15.23} \]

\[ i(t) = \frac{1}{18} \frac{d v}{dt} \quad \text{Eq. 15.23a} \]

\[ \frac{15}{18} \frac{d v}{dt} + \frac{3}{18} \frac{d^2 v}{dt^2} + v(t) = 25 \quad \text{Eq. 15.24} \]
\[
\frac{\partial^2 V(t)}{\partial t^2} + 5 \frac{\partial V(t)}{\partial t} + 6 V(t) = 0
\]  
Eq 15.25

The particular solution is
\[V_p(t) = 25\]

For the transient solution, we must find the roots from the characteristic equation:
\[s^2 + 5s + 6 = (s+2)(s+3) = 0\]
Thus
\[V(t) = Ae^{-2t} + Be^{-3t}\]  
Eq 15.26

Then
\[V(t) = 25 + Ae^{-2t} + Be^{-3t}\]  
Eq 15.27

We now evaluate \(A\) & \(B\) using initial conditions. We know that
\[V(0^-) = V(0^+) = 0\]  
Eq 15.28

\[V(0^-) = 25 = V(0^+)\]

We need \(V(0^+), \frac{\partial V(t)}{\partial t}\) to use in
Equation 15.27 to solve for \(A\) & \(B\).

From Eq 15.23A we have
\[\frac{\partial V(0)}{\partial t} = 180V(0) = 0\]  
Eq 15.29

We can now use position to solve for \(A\) & \(B\) in Eq 15.27
We have

\[ V(0) = 30 \leq 25 + AE^{-2t} + BE^{-3t} \]

or

\[ A + B = 10 \]  \hspace{1cm} \text{Eq. 15.30} \]

\[ \frac{\partial V}{\partial t} = -2AE^{-2t} - 3BE^{-3t} \]

or

\[ \frac{\partial V(0)}{\partial t} = 0 = -2A - 3B \]  \hspace{1cm} \text{Eq. 15.31} \]

\[ A + B = 10 \]

\[ 2A + 3B = 0 \]

Which gives:

\[ A = 30, \ B = -20 \]

Then

\[ V(t) = 25 + 30e^{-2t} - 20e^{-3t} \]  \hspace{1cm} \text{Eq. 15.32} \]

Example 15.3:

The following problem is taken from "Basic Engineering Circuit Analysis", J. David Irwin, John Wiley & Sons, pp 226.

The switch in the network of Figure 15.11 has been in position 1 for a very long time. At \( t = 0 \), the switch is changed to position 2. Find \( V_0(t) \) for \( t \geq 0 \).
Figure 15.11; Circuit for Example 15.3.

For $t < 0$, we note $i'(0) = 2$ Amps.

Since there is no resistance in the branch with the inductor, $V_{ab} = 0$. Also, the voltage across the capacitor is zero since no current is flowing through the $2\Omega$ resistor, hence $V_b(0) = 0$. and since $V_{ab}(0) = 0$, we know the capacitor voltage at $t=0$ is zero.

For $t > 0$.

Figure 15.12; Circuit for example 15.3 for $t > 0$. 
We can write:

\[ R \frac{d^2 i(t)}{dt^2} + L \frac{di(t)}{dt} + \frac{1}{C} \int_{0}^{t} i(t') dt' = 0 \quad \text{Eq. 15.33} \]

for the circuit of Figure 15.12.

Taking the derivative of this equation gives

\[ L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = 0 \quad \text{Eq. 15.34} \]

Substituting numerical values gives

\[ \frac{d^2 i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0 \quad \text{Eq. 15.35} \]

The characteristic equation is:

\[ s^2 + 4s + 3 = (s+3)(s+1) = 0 \]

Then

\[ i(t) = Ae^{-3t} + Be^{-t} \quad \text{Eq. 15.35} \]

We need

\[ i(0), \quad \frac{di(0)}{dt} \]

for initial conditions. We have

\[ i(0) = 2A. \quad \text{From Eq. 15.33 we have} \]

\[ \frac{d^2 i(t)}{dt^2} = -\frac{R}{L} \int_{0}^{t} i(t') dt' + \frac{1}{LC} \]

\[ \text{Eq. 15.36} \]
\[ \frac{\partial i(t)}{\partial t} = -4 \frac{i(t)}{\partial t} = -8 \]

Then from Eq. 15.35:

\[ i(0) = 2 = A e^{-3 \cdot 0} + B e^{-8} = A + B \]

\[ A + B = 2 \]  

and

\[ \frac{\partial i(t)}{\partial t} = -3 A e^{-3t} - B e^{-t} \]

\[ 3A + B = 8 \]  

Solving Eq. 15.37 and 15.38 gives

\[ A = 3, \ B = -1 \]

\[ i(t) = 3 e^{-3t} - e^{-t} \]  

Eq. 15.39

Now

\[ V_o(t) = -2 \frac{\partial i(t)}{\partial t} \]

\[ V_o(t) = 2 (e^{-t} - 3e^{-3t}) \]  

Eq. 15.40
Example 15.4

You are given the following circuit.

Figure 15.13: Circuit for Example 15.4.

All initial conditions are zero.

\( R = 5 \Omega, \quad C = 0.04 \text{ F}, \quad L = 1 \text{ H} \)

We will use nodal analysis to set up the differential equation. We have

\[ i_R(t) + \dot{i}_C(t) + \dot{i}_L(t) = 0 \]

dr

or

\[ \frac{V_{01}(t)}{R} + C \frac{dV_{01}(t)}{dt} + i_L(t) = 0 \quad \text{Eq. 15.4} \]

Since

\[ V_L(t) = L \frac{di_L}{dt} \]

\[ i_L(t) = \frac{1}{L} \int_0^t V_L(\tau) d\tau + i_L(0) \quad \text{Eq. 15.42} \]
We note by KVL that
\[ V_o(t) - V_L(t) - 1 = 0 \]

or
\[ V_L(t) = V_o(t) - 1 \]

Eq. 15.43

With \( i_L(0) = 0 \), we substitute the above into Eq. 15.42 to find
\[ i_L(t) = \frac{1}{L} \int_0^t (V_o(\tau) - 1) \, d\tau \]

Eq. 15.44

Now substitute Eq. 15.44 into Eq. 15.41. This gives
\[
\frac{V_o(t)}{R} + C \frac{d^2 V_o(t)}{dt^2} + \frac{1}{L} \int_0^t (V_o(\tau) - 1) \, d\tau = 0
\]

We take the derivative of the above with respect to \( t \):
\[
\frac{1}{R} \frac{dV_o(t)}{dt} + C \frac{d^2 V_o(t)}{dt^2} + \frac{(V_o(t) - 1)}{L} = 0
\]

Eq. 15.45

Or
\[
\frac{d^2 V_o(t)}{dt^2} + \frac{1}{RC} \frac{dV_o}{dt} + \frac{V_o(t)}{LC} = \frac{1}{LC}
\]

We now put in numerical values to find the 2nd order diff. equation.
\[ \frac{d^2 V_0(t)}{dt^2} + 5 \frac{dV_0}{dt} + 25V_0(t) = 0 \quad \text{Eq. 15.46} \]

We compare this to Eq. 15.6 for \( V_0(t) \) rather than \( x(t) \) and we have

\[ \frac{d^2 V_0(t)}{dt^2} + 2 \zeta \omega_n \frac{dV_0}{dt} + \omega_n^2 V_0(t) = 0 \quad \text{Eq. 15.47} \]

We have already solved the above equation for zero initial conditions. This was Eq. 15.21 (p. 15.15) which is repeated below:

\[ V_0(t) = 1 + \frac{e}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t + \theta) \quad \text{Eq. 15.48} \]

Comparing Equation 15.46 and 15.47 gives

\[ \omega_n = 5 \quad \zeta = 0.5 \]

Substituting into Eq. 15.48 gives

\[ V_0(t) = 1 + 1.16 e^{-2.5t} \sin(4.33t + 60^\circ) \quad \text{Eq. 15.49} \]
If we want to sketch this waveform we recall that from Eq. 15.21 (p. 15.16)
\[ T_p = \frac{\pi}{\ln(1-\delta^2)} \quad \text{Eq. 15.21} \]
and
\[ D.S. = 100 \left( 1 - e^{-\frac{4\pi}{\ln(1-\delta^2)}} \right) \% \quad \text{Eq. 15.22} \]
Using our value of \( \delta \) and \( \zeta \) we have
\[ T_p = 0.73 \text{ sec} \quad D.S. = 16.3\% \]
so our response can now be sketched.

One can easily use MATLAB to verify this response.