Passivity-Based Position Consensus of Multiple Mechanical Integrators with Communication Delay

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Abstract—We present a consensus framework for multiple variable-rate heterogeneous mechanical integrators (i.e. multi-dimensional integrators with mass, damping and spring matrices) on undirected graph with constant, yet, non-uniform communication delay. By connecting multiple mechanical integrators via (discrete-time) spring connections over delayed links with some damping injection, our proposed framework not only achieves position consensus, but also enforces closed-loop discrete-time passivity. Moreover, it allows arbitrary control gains regardless of integration steps, if there is no delay. Simulation result is also given to support the theory.

I. INTRODUCTION

Multiagent consensus problem has been an active research area, due to its implications across many application domains and disciplines: multirobot coordination [1], sensor network [2], flocking and swarming in nature and animation [3], distributed computing [4], human collective behavior [5], to name a few. Many strong results have been reported for both the continuous-time and discrete-time first-order systems (e.g. [6], [7], [8]). Relatively rare results have been reported for the second-order systems (e.g. [9], [10]), with even lesser results available for discrete-time second-order consensus with communication delays (e.g. [11], [12], [13]).

In this paper, we present a new consensus framework for multiple mechanical integrators (i.e. multi-dimensional second-order discrete-time integrator with mass, damping and spring matrices) on undirected graph with constant, yet, non-uniform inter-agent communication delays. One of the unique properties of our proposed framework is that, by connecting the mechanical integrators (MI) via (discrete-time) spring connections over delayed communication links with some damping injection, it not only achieves asymptotic position consensus among MIs, but also enforces discrete-time passivity of the closed-loop multiple MIs.

For this, we first utilize our recently proposed non-iterative variable-rate passive MI of [14], which is discrete-time passive unlike explicit integrators used in other works (i.e. update depends solely on the values of previous cycle), yet, can still be simulated fast enough (particularly for haptics) due to its non-iterativeness. We then extend the PD (proportional-derivative) scheme of [15] into the discrete-time domain to enforce closed-loop passivity and position consensus of the multiple MIs over undirected communication graph with non-uniform constant delays.

The motivating application of this work is peer-to-peer shared multiuser haptic interaction over packet-switching communication network (e.g. Internet with some data buffering), where distributed users haptically interact with their own MI (i.e. local copy of a shared virtual environment) via their haptic device, while these MIs are being synchronized to provide consistent perception of the shared virtual environment among the users. For this, the aforementioned closed-loop passivity is very useful, since it would allow us to ensure interaction stability with wide-ranges of (continuous-time) heterogeneous/unmodeled haptic devices or human users, as long as they behave as passive systems. See [16].

To our knowledge, none of results so far on discrete-time second-order consensus (e.g. [11], [12], [13]) has achieved this closed-loop passivity with delay. In addition to its unique passivity property, our framework here also can handle with: non-uniform delay (unlike [11], [12]); multi-dimensionality (unlike [13]); and variable update rate and non-uniformity among the MIs (unlike [11], [12], [13]). Our framework also allows arbitrary gains regardless of sampling rate, if there is no delay (unlike [11], [12]).

The rest of the paper is organized as follows. We review some related graph theory and non-iterative variable-rate passive MI of [14], and also introduce our problem setting in Sec. II. The consensus of multiple MIs is then discussed with a numerical example in Sec. III, and some concluding remarks are given in Sec. IV.

II. BACKGROUND

A. Graph Theory

In this paper, we consider undirected and simple (i.e. no self-loop/multiple-edges) graph $G = (V, E)$, where $V := \{v_1, \ldots, v_N\}$ is the set of vertexes; and $E$ refers to the set of edges. If $v_i$ and $v_j$ are connected, we say that the edge from $v_i$ to $v_j$ (denote as $e_{ij}$) belongs to $E$. The set of information neighbors of a vertex $v_i$ is defined as $N_i := \{j|e_{ij} \in E\}$. If $i \in \mathcal{S} \subseteq \mathcal{E}$ is undirected, $e_{ij} \in \mathcal{E}$ implies $e_{ji} \in \mathcal{E}$.

Suppose the total number of (directed) edges $|E|$ of $G$ is $N_{v}$. Then, there exists a bijective map $\mathcal{F}: \mathcal{S} \rightarrow \mathcal{E}$ s.t.

$$\mathcal{F}(l) = (p_l, q_l)$$

where $\mathcal{S} := \{1, \ldots, N_{v}\}$ and, with a slight abuse of notation, we denote the set of $\{(i,j)|e_{ij} \in \mathcal{E}\}$ by the edge set $\mathcal{E}$. For simplicity, we will often omit subscript $l$ for $(p_l, q_l)$.
The following two properties will be used with their proof omitted here. ∀e_{ij} \in \mathbb{R}^{n \times n}
\sum_{i=1}^{N} \sum_{j \in N_i} e_{ij} = \sum_{i=1}^{N} e_{pq}, \quad \forall e_{pq} = \frac{1}{2} \sum_{i=1}^{N} (e_{pq} + e_{qp})

and further, if G(V, E) is undirected,
\sum_{i=1}^{N} e_{pq} = \frac{1}{2} \sum_{i=1}^{N} (e_{pq} + e_{qp})

where (p, q) = \mathcal{F}(i).

The incidence matrix \mathcal{D} = \{d_{ij}\} \in \mathbb{R}^{N \times N} of G is defined by
\begin{cases}
1 & \text{if } v_i \text{ is the initial end of the connection } e_l \\
-1 & \text{if } v_i \text{ is the terminal end of the connection } e_l \\
0 & \text{otherwise}
\end{cases}

while the Laplacian matrix \mathcal{L} = \{l_{ij}\} \in \mathbb{R}^{N \times N} of G is given by
\begin{cases}
\deg(v_i) & \text{if } i = j \\
-1 & \text{if } e_{ij} \in \mathcal{E} \\
0 & \text{otherwise}
\end{cases}

where \deg(v_i) refers to the degree (number of edges) of v_i.

We then have [17, Prop.4.8]:
\mathcal{L} = \mathcal{D} \mathcal{D}^T

It is well-known that, if G is undirected and connected, the zero eigenvalue of \mathcal{L} is simple with the eigenvector \mathbf{1} := [1, \ldots, 1]^T \in \mathbb{R}^N, and all the other eigenvalues have strictly positive real parts [6, 7, 17].

B. Non-Iterative Variable-Rate Passive MIs

In this paper, we utilize the non-iterative variable-rate passive mechanical integrators (MIs) proposed in [14]: for a single mass-spring-damper type MI, during the integration step \( T_k > 0 \),
\begin{align}
M \frac{v_{k+1} - v_k}{T_k} + B \frac{v_{k+1} + v_k}{2} + K \frac{x_{k+1} + x_k}{2} &= \tau_k + f_k \\
\frac{v_{k+1} - v_k}{T_k} &= \frac{x_{k+1} - x_k}{T_k}
\end{align}

where \( M, B, K \in \mathbb{R}^{n \times n} \) are respectively mass, damper, and spring matrices (with \( M, K \) being symmetric/positive-definite, and \( B \) being positive-semidefinite); \( x_k, v_k \in \mathbb{R}^n \) is the position/velocity; and \( \tau_k, f_k \in \mathbb{R}^n \) are the control and human/environmental force respectively. Here, the first line is the dynamics update; while the second kinematics.

This integrator is implicit (i.e. the future state does not only depend on the previous one but also on itself too), yet, still non-iterative (i.e. both \( x_{k+1} \) and \( v_{k+1} \) can be expressed as functions depending on \( x_k \) and \( v_k \) explicitly, so they can be solved without iterations), thus, can still be simulated fast. The main reason of using this MI over other integrators (e.g. integrators used in [11], [12], [13], [18]) is that it possesses the following open-loop discrete passivity:
\begin{align}
\sum_{k=0}^{\hat{M}} \hat{\psi}_k^T (\tau_k + f_k) T_k &\geq v_{k+1} - v_0 \geq -V_o \quad \forall \hat{M} \geq 0
\end{align}

where \( \hat{\psi}_k := (v_{k+1} + v_k)/2 \) and \( V_k := v_k^T M v_k/2 + x_k^T K x_k/2 \) is the total energy with \( V_o \) being the initial energy. This discrete-time passivity allows us to connect this MI with a variety of haptic devices or human users while ensuring interaction stability, provided that they behave as passive systems. In this paper, we will utilize and also aim to preserve this passivity of the MIs. See [14] for more details on this non-iterative passive integrator.

C. Consensus Setting

We consider \( N \) MIs on an undirected communication graph \( G(V, E) \), where each edge \( e_{ij} \in \mathcal{E} \) (i.e. from \( i^{th} \)-MI to \( j^{th} \)-MI) is associated with constant indexing delay \( N_{ij} \geq 0 \) and a symmetric/positive-definite spring matrix \( K_{ij} \in \mathbb{R}^{n \times n} \). See Fig. 1. Then, following Sec. II-B, we write down the following consensus protocol for the \( i^{th} \)-MI: during the update interval \( T_i(k) > 0 \),
\begin{align}
M_i \frac{v_i(k+1) - v_i(k)}{T_i(k)} &= \tau_i(k) + f_i(k) \quad \forall \hat{M} \geq 0
\end{align}

where \( \hat{\psi}_k := (v_{k+1} + v_k)/2 \) and \( \hat{\psi}_k := (x_{k+1} + x_k)/2; \) \( N_i \) is the information neighbors of the \( i^{th} \)-MI; \( B_i \in \mathbb{R}^{n \times n} \) is the damping matrix of \( i^{th} \)-MI; \( K_{ij} \in \mathbb{R}^{n \times n} \) is the spring matrix associated with \( e_{ij} \); \( N_{ij} \geq 0 \) is the constant indexing delay from \( j^{th} \)-MI to \( i^{th} \)-MI; and \( f_i(k) \) is the force acting on \( i^{th} \)-MI (e.g. virtual coupling/contact with an virtual object). For the spring connection \( K_{ij} \), we assume
\begin{align}
K_{ij} &= K_{ji}
\end{align}

i.e. symmetric (yet, still can be non-uniform) spring connection, although their delays may not be so (i.e. \( N_{ij} \neq N_{ji} \)). Note that the update rate \( T_i(k) \) can be varying. With this variable \( T_i(k) \), the constant indexing delay \( N_{ij} \) does not necessarily imply constant time delay.

III. Consensus of Multiple Mechanical Integrators on Delayed Undirected Graph

As mentioned in Sec. II-C, the spring connections in the system are multi-dimensional and non-uniform (i.e. the spring matrix \( K_{ij} \) can vary from connection to connection). These features cannot be captured by the graph Laplacian \( \mathcal{L} \) due to its definition (3). Therefore, we will need to use the
following stiffness matrix, instead of the graph Laplacian $L$, as the tool to show the position consensus among $N$ MIs connected via $K_l^i$, with each other. The stiffness matrix $P \in \mathbb{R}^{N \times N}$ with its $i$-th block matrices $P_{ij} \in \mathbb{R}^{n \times n}$ is given as

$$P_{ij} = \begin{cases} K_{ij} & i = j \text{ and } e_{ij} \in E \\ -K_{ij} & i \neq j \text{ and } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

where $K_{ij} \in \mathbb{R}^{n \times n}$ is the spring matrix associated with $e_{ij}$. The next proposition shows that the stiffness matrix has close relation with incidence matrix on an undirected graph.

**Proposition 1:** Consider $N$ MIs on an undirected communication graph $G(V, E)$. The stiffness matrix as defined in (10) can be represented as,

$$P = \frac{1}{2} \left( (\mathbb{D} \otimes I_n) K_d (\mathbb{D} \otimes I_n) \right)^T \quad (11)$$

where $\mathbb{D}$ is the incidence matrix of $G(V, E)$ (defined in Sec. II), $\otimes$ refers to the Kronecker product, and $K_d := \text{diag}(K_1, K_2, \ldots, K_{N_k})$ with $K_i$ the spring matrix associated with $e_l$, $l \in S = \{1, \ldots, N_k\}$.

**Proof:** Let us define $\hat{D} := (\mathbb{D} \otimes I_n) K_d^{1/2} \in \mathbb{R}^{N \times nN}$. Then, its $i$-th block matrix $\hat{d}_{il} \in \mathbb{R}^{n \times n}$ is given by

$$\hat{d}_{il} = \begin{cases} K_{il}^{1/2} & \text{if } v_i \text{ is the initial end of the edge } e_l \\ -K_{il}^{1/2} & \text{if } v_i \text{ is the terminal end of the edge } e_l \\ 0 & \text{otherwise} \end{cases}$$

where $i$ represents the index of MI; while $K_i$ is the spring matrix associated with the edge $e_l$, $l \in S$. Let us denote the edges connecting $k$-th MI by $\hat{N}_k := \{ r | e_r \text{ is incoming or outgoing edge of } v_k \}$. Then, $\hat{d}_{ik} = \pm K_{il}^{1/2} \neq 0_n$ ($n \times n$ zero matrix), if $l \in \hat{N}_k$; and $\hat{d}_{il} = 0_n$ otherwise.

Now, we are going to show that $\hat{D} \hat{D}^T = 2P$. First, note that, the $k$-th $n \times n$ diagonal block of $\hat{D} \hat{D}^T$ is given by

$$\sum_{l=1}^{N_k} \hat{d}_{kl}^2 = \sum_{l \in \hat{N}_k} K_l = 2 \sum_{j \in \hat{N}_j} K_{lj}$$

where the last equality is because $G(V, E)$ is undirected and $\hat{N}_k$ includes both the incoming and outgoing edges of $v_k$. Also, for the off-diagonal block matrices of $\hat{D} \hat{D}^T$, we have, for $i \neq j$

$$\sum_{l=1}^{N_k} \hat{d}_{il} \hat{d}_{jl} = \sum_{l \in \hat{N}_i \cap \hat{N}_j} \hat{d}_{il} \hat{d}_{jl} \quad (12)$$

since, for $\hat{d}_{il} \hat{d}_{jl} \neq 0_n$, it requires $l \in \hat{N}_i \cap \hat{N}_j$. Suppose $e_a, e_b$ are the 1-tuple equivalence of $e_{ij}, e_{ji} \in E$ respectively (i.e. $\mathcal{F}(a) = (i, j), \mathcal{F}(b) = (j, i)$). Then, we have $\hat{N}_i \cap \hat{N}_j = \{a, b\}$. And (12) will become $- \sum_{K_l + K_b} = -(K_{il} + K_{jl}) = -2K_{ij}$ from the definition of $\hat{d}_{il}$ above and (9). By comparing these diagonal and off-diagonal block matrices of $\hat{D} \hat{D}^T$ with $P$ in (11), we can then see that

$$P = \frac{1}{2} \hat{D} \hat{D}^T$$

One of the strong properties of the graph Laplacian $L$ is: for a connected undirected graph $G(V, E)$ with $N$ nodes, $\text{rank}(L) = N - 1$. The following lemma extends this property to stiffness matrix.

**Lemma 1:** For a simple undirected graph $G(V, E)$, we have

$$\text{rank}(P) = n \text{ rank}(L)$$

where $n$ is dimension of MI; $L$ is graph Laplacian of $G(V, E)$; and $P$ is the stiffness matrix in (10). Moreover, if $G(V, E)$ is connected,

$$\text{rank}(P) = n(N - 1)$$

**Proof:** By Proposition 1, the properties of rank and Kronecker product, and (4), we have,

$$\text{rank}(P) = \text{rank} \left( [(\mathbb{D} \otimes I_n) K_d^{1/2}] [(\mathbb{D} \otimes I_n) K_d^{1/2}]^T \right)$$

$$= \text{rank}(K_d^{1/2} \mathbb{D} \otimes I_n K_d^{1/2})$$

$$= \text{rank}(I_n \text{rank}(\mathbb{D} \otimes I_n))$$

$$= n \text{ rank}(L)$$

since $K_d$ is nonsingular. Also, if $G(V, E)$ is connected, we have $\text{rank}(L) = N - 1$. Thus,

$$\text{rank}(P) = n(N - 1)$$

For the $N$ MIs system with the update law (6)-(7), we define the close-loop discrete-time passivity s.t. $\forall \bar{M} \geq 0$

$$\sum_{i=1}^{N} \sum_{k=0}^{\bar{M}} \bar{v}_i(k)^T f_i(k) T_i(k) \geq -c^2 \quad (13)$$

where $c \in \mathbb{R}$ is a bounded constant. Following [15], [19], [20], we also define the controller passivity s.t. $\forall \bar{M} \geq 0$

$$\sum_{i=1}^{N} \sum_{k=0}^{\bar{M}} \bar{v}_i(k)^T \tau_i(k) T_i(k) \leq d^2 \quad (14)$$

where $d \in \mathbb{R}$ is a bounded constant.

The following lemma shows the connection between closed-loop discrete-time passivity and controller passivity.

**Lemma 2:** For the $N$ MIs system with the update law (6)-(7), controller passivity (14) implies the closed-loop discrete-time passivity (13).

**Proof:** By (6), (7), and (14),

$$\sum_{i=1}^{N} \sum_{k=0}^{\bar{M}} \bar{v}_i(k)^T f_i(k) T_i(k)$$

$$= \sum_{i=1}^{N} \sum_{k=0}^{\bar{M}} \bar{v}_i(k)^T \left[ M_i v_i(k+1) - v_i(k) \right] T_i(k)$$

$$\geq -\frac{\bar{M}}{2} \sum_{i=1}^{N} ||v_i(0)||_{M_i}^2 - d^2 =: -c^2 \quad (15)$$

where $\bar{M} \sum_{i=1}^{N} ||v_i(0)||_{M_i}^2$ represents the initial kinetic energy inside the system, with the notation $||x||_A^2 := x^T A x$ ($A \in \mathbb{R}^{n \times n}$ is required to be symmetric and positive-definite).
As mentioned in Sec. I, closed-loop discrete-time passivity and position consensus among MI’s ensure the interaction stability and consistent perception of the shared virtual environment among the users. The following theorem shows that, with enough damping injection, our consensus protocol (6)-(8) achieves both the closed-loop discrete-time passivity and position consensus.

**Theorem 1:** Consider $N$ MI’s on an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with the consensus protocol (6)-(8). Suppose that we set the gains $B_i, K_{ij}$ such that

$$B_i \geq \sum_{j \in N_i} \frac{N_{ij} + N_{ji}}{2} \bar{T}_{ij}, \quad i = 1, \ldots, N \tag{16}$$

where $\bar{T}_{ij} = \max_\alpha(T_{ij}(\alpha))$, and $v(k) = 0, \forall k \leq 0$. Then,

1. the system achieves controller passivity (14), thus, also closed-loop passivity (13); and
2. if there is additional positive-definite damping $B_p^2 \in \mathbb{R}^{n \times n}$ on each MI, and $f_i(k) = 0$, the position consensus is achieved.

$$x_i(k) \rightarrow x_{ij}(k) \quad \text{and} \quad v_i(k) \rightarrow 0, \quad \forall i, j = 1, \ldots, N.$$

**Proof:** The controller passivity condition (14) means that the controller only generates bounded energy the whole time. Hence we can prove the first item by showing this energy boundedness. Denote the energy generated by the controller during $k$th integration step as

$$s_E(k) := \sum_{i=1}^{N} \hat{v}_i(k)^T \tau_i(k) T_i(k)$$

$$= \sum_{i=1}^{N} \left[ \sum_{j \in N_i} -D_i(k) - \hat{v}_i(k)^T K_{ij} (\hat{x}_j(k) - \hat{x}_j(k - N_{ij})) \right] T_i(k)$$

$$= -\sum_{i=1}^{N} D_i(k) - \sum_{i=1}^{N} \hat{v}_i(k)^T K_{pq} \left[ \hat{x}_p(k) - \hat{x}_q(k - N_{pq}) \right] T_p(k)$$

where $D_i(k) := \|\hat{v}_i(k)\|^2 T_i(k)$, and $(p, q) = \mathcal{F}(I)$. The last term can then be rewritten as

$$\sum_{i=1}^{N} \left[ \hat{v}_i(k)^T K_{pq} \left[ \hat{x}_q(k) - \hat{x}_q(k - N_{pq}) \right] T_p(k) \right]$$

$$= -\sum_{i=1}^{N} \hat{v}_i(k)^T K_{pq} \left[ \hat{x}_p(k) - \hat{x}_q(k - N_{pq}) \right] T_p(k)$$

Then, we can rewrite $s_E(k)$ s.t.

$$s_E(k) = \sum_{i=1}^{N} \left[ \phi_{pq}(k) - \phi_{pq}(k + 1) \right] - \sum_{i=1}^{N} D_i(k)$$

$$- \sum_{i=1}^{N} \hat{v}_i(k)^T K_{pq} \left[ \hat{x}_q(k) - \hat{x}_q(k - N_{pq}) \right] T_p(k) \tag{18}$$

Let us define

$$\Theta_{pq}(k) := \hat{v}_p(k)^T K_{pq} \left[ \hat{x}_q(k) - \hat{x}_q(k - N_{pq}) \right] T_p(k)$$

By inserting intermediate terms $\sum_{j=k+1-N_{pq}}^{k+1} \left( -\hat{x}_q(j) + \hat{x}_q(j) \right)$ between $\hat{x}_q(k)$ and $\hat{x}_q(k - N_{pq})$, we can then rewrite $\Theta_{pq}(k)$ as

$$\Theta_{pq}(k) = T_p(k) \hat{v}_p(k)^T K_{pq} \sum_{j=k+1-N_{pq}}^{k+1} \left( -\hat{x}_q(j + 1) + \hat{x}_q(j) \right)$$

$$= T_p(k) \hat{v}_p(k)^T K_{pq} \left[ \sum_{j=k+1-N_{pq}}^{k+1} \hat{v}_q(j + 1) \right] T_q(k) + \hat{v}_q(j) T_q(k) \right]$$

$$= T_p(k) \hat{v}_p(k)^T K_{pq} \left[ \sum_{j=k+1-N_{pq}}^{k+1} \hat{v}_q(j) T_q(k) \right]$$

$$+ \frac{1}{2} \left( \hat{v}_q(k) T_q(k) + \hat{v}_q(k - N_{pq}) T_q(k - N_{pq}) \right)$$

where the second line is due to (7). Since $K_{pq}$ is symmetric and positive-definite, we have the following fact s.t.

$$|a^T K_{pq} b| \leq \frac{1}{2} \left( |a| \|K_{pq}\| + |b| \|K_{pq}\| \right), \quad \forall a, b \in \mathbb{R}^n$$

Using this, we can then show that

$$\Theta_{pq}(k) \leq \frac{1}{2} T_p(k) \sum_{j=k+1-N_{pq}}^{k+1} T_q(k) \left[ \|\hat{v}_p(\mathcal{K})\|^2 + \|\hat{v}_q(\mathcal{K})\|^2 \right]$$

$$+ \frac{1}{4} T_p(k) T_q(k) \left[ \|\hat{v}_p(\mathcal{K})\|^2 + \|\hat{v}_q(\mathcal{K})\|^2 \right]$$

$$+ \frac{1}{4} T_p(k) T_q(k) \left[ \|\hat{v}_p(\mathcal{K})\|^2 + \|\hat{v}_q(\mathcal{K})\|^2 \right]$$

$$= \frac{1}{4} \alpha_{pq}(k) \left[ T_q(k) + T_q(k - N_{pq}) + 2 \sum_{j=k+1-N_{pq}}^{k+1} T_q(j) \right]$$

$$+ \frac{1}{4} T_p(k) \left[ \alpha_{pq}(k) + \alpha_{pq}(k - N_{pq}) + \sum_{j=k+1-N_{pq}}^{k+1} \alpha_{pq}(j) \right]$$

where $\alpha_{pq}(k) := T_q(k) \|\hat{v}_q(\mathcal{K})\|^2 \|K_{pq}\| \geq 0$. Since $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is undirected, for each $\Theta_{pq}(k)$, we will also have the corresponding $\Theta_{qp}(k)$. By summing them up and collecting the terms containing $\alpha_{pq}$ and $\alpha_{qp}$ respectively, we can show that

$$|\Theta_{pq}(k)| + |\Theta_{qp}(k)| \leq \Omega_{pq}(k) + \Omega_{qp}(k)$$

where

$$\Omega_{pq}(k) := \frac{1}{2} T_q(k) \left[ \frac{1}{2} \alpha_{pq}(k - N_{pq}) + \sum_{j=k+1-N_{pq}}^{k+1} \alpha_{pq}(j) \right]$$

$$+ \frac{1}{2} \alpha_{pq}(k) \left[ T_q(k) + \frac{1}{2} T_q(k - N_{pq}) + \sum_{j=k+1-N_{pq}}^{k+1} T_q(j) \right]$$
Then, by summing $\Omega_{pq}(k)$ over time, we have
\[
\sum_{k=0}^{\bar{M}} \Omega_{pq}(k) = \\
\frac{1}{2} \sum_{k=0}^{\bar{M}} \alpha_{pq}(k) \left[ T_q(k) + \frac{1}{2} T_q(k - N_{pq}) + \sum_{j=k+1-N_{pq}}^{k-1} T_q(j) \right] + \frac{1}{4} \sum_{k=0}^{\bar{M}} \alpha_{pq}(k-N_{qp}) T_q(k) + \frac{1}{2} \sum_{k=0}^{\bar{M}} \alpha_{pq}(k) \sum_{j=k+1-N_{pq}}^{k-1} \alpha_{pq}(j) + \frac{1}{4} \sum_{k=0}^{\bar{M}} \alpha_{pq}(k) T_q(k + N_{qp}) - \sum_{k=M+1-N_{pq}}^{\bar{M}} \alpha_{pq}(k) T_q(k + N_{qp}) \right] + \frac{1}{4} \sum_{k=0}^{\bar{M}} \alpha_{pq}(k) T_q(j) \sum_{j=M+1-N_{pq}}^{N_{qp}-2} \alpha_{pq}(M-k) \sum_{j=k+1-N_{pq}}^{M-1} T_q(j) \right) \tag{*} \*
\leq \sum_{k=0}^{\bar{M}} \frac{1}{4} \alpha_{pq}(k) \left[ T_q(k - N_{pq}) + T_q(k + N_{qp}) + 2 \sum_{j=k+1-N_{pq}}^{k-1} T_q(j) \right] \tag{**}
\right)
\]
where the last term in (**) comes from the $1^{st}$, $3^{rd}$ and $6^{th}$ terms in (*). From (**), with $\alpha_{pq}(k) \geq 0$ and $T_q \geq T_q(k)$, we have
\[
\sum_{k=0}^{\bar{M}} \Omega_{pq}(k) \leq \frac{N_{pq} + N_{qp} \bar{T}_q \sum_{k=0}^{\bar{M}} \alpha_{pq}(k)}{2}
\]
Then, we can rewrite (18) s.t.: $\forall M \geq 0,
\sum_{k=0}^{\bar{M}} \sum_{k=0}^{N} p \sum_{i=1}^{N} \psi_i^T(k) \tau_i(k) T_i(k) \\
\leq \sum_{i=1}^{N} \phi_{pq}(0) - \sum_{k=0}^{\bar{M}} \sum_{p=1}^{N} D_p(k) + \sum_{k=0}^{\bar{M}} \sum_{p=1}^{N} \sum_{q \in N_p} |\Theta_{pq}(k)| \\
\leq \sum_{i=1}^{N} \phi_{pq}(0) - \sum_{k=0}^{\bar{M}} \sum_{p=1}^{N} D_p(k) + \sum_{k=0}^{\bar{M}} \sum_{p=1}^{N} \sum_{q \in N_p} \Omega_{pq}(k) \\
\leq \sum_{i=1}^{N} \phi_{pq}(0) =: d^2 \tag{19} \]
where $d$ is a bounded constant, and we use the fact that under the gain-setting condition (16),
\[
\sum_{k=0}^{\bar{M}} \sum_{q \in N_p} \Omega_{pq}(k) - \sum_{k=0}^{\bar{M}} D_p(k) \\
\leq \sum_{q \in N_p} \frac{N_{pq} + N_{qp}}{2} \bar{T}_q \sum_{k=0}^{\bar{M}} \alpha_{pq}(k) - \sum_{k=0}^{\bar{M}} D_p(k) \\
= \sum_{k=0}^{\bar{M}} \psi(k)^T \left[ \sum_{q \in N_p} \frac{N_{pq} + N_{qp}}{2} \bar{T}_q K_{pq} - B_p \right] \psi(k) T_p(k) \leq 0
\]
where the third line is due to (16). This proves the controller passivity (14). Thus, by Lemma 2, closed-loop passivity (13) is achieved.

For the second item, from the closed-loop passivity (13), with the additional $B_i^T$ and $f_i(k) = 0$, we have: $\forall \bar{M} > 0$
\[
\mathcal{P}^2 \geq \sum_{k=0}^{\bar{M}} \sum_{i=1}^{N} \psi_i(k) \|^2 \mathcal{P} \tag{20}
\]
which implies that, with $c$ being bounded, $\psi_i(k) \to 0$. With this, (6) at $k^{\text{th}}$ and $k+1^{\text{th}}$ integration steps are reduced to
\[
M_i \frac{v_i(k + 1) - v_i(k)}{T_i(k)} \to - \sum_{j \in N_i} K_{ij} (\hat{x}_i(k) - \hat{x}_j(k - N_{ij})) \tag{21} \]
\[
M_i \frac{v_i(k + 2) - v_i(k + 1)}{T_i(k + 1)} \to - \sum_{j \in N_i} K_{ij} (\hat{x}_i(k + 1) - \hat{x}_j(k + 1 - N_{ij})) \tag{22} \]
Also, from (7) and $\hat{x}_i(k) \to 0$, $x_i(k) \to x_i(k - 1)$ and $x_i(k) \to \hat{x}_i(k - 1) \to \hat{x}_i(k - 2) \to \cdots \to \hat{x}_i(k - N_{ij})$. Therefore, the spring forces of the two successive time steps converge to each other, i.e.
\[
\sum_{j \in N_i} K_{ij} (\hat{x}_i(k) - \hat{x}_j(k - N_{ij})) \to 0
\]
thus, combining this with (21)-(22), we have
\[
(v_i(k + 1) - v_i(k)) T_i(k + 1) \to (v_i(k + 2) - v_i(k + 1)) T_i(k)
\]
where, from $\hat{x}_i(k) \to 0$, $v_i(k + 2) \to v_i(k)$. Therefore,
\[
(v_i(k + 1) - v_i(k)) T_i(k + 1) \to - (v_i(k + 1) - v_i(k)) T_i(k)
\]
implying that, with $T_i(k) > 0$, $v_i(k + 1) \to v_i$. Moreover, with $\hat{x}_i(k) \to 0$, we have $v_i(k) \to 0$. Substituting this into (6) with $f_i(k) = 0$, we have
\[
\sum_{j \in N_i} K_{ij} (x_i(k) - x_j(k)) \to 0, \quad i = 1, \ldots N
\]
which can be written by
\[
\mathcal{P} x(k) \to 0 \tag{23}
\]
where $\mathcal{P}$ is the stiffness matrix defined in (10), and $x(k) := [x_1(k)^T \ldots x_N(k)^T]^T$. This (23) then implies
\[
x(k) \to 1 \odot d = [d^T \ldots d^T]^T
\]
because: 1) by Lemma 1, the dimension of $\text{ker} (\mathcal{P})$ is $n$. 2) $\mathcal{P} (1 \odot d) = 0, \forall d \in \mathbb{R}^n$ (i.e. $\text{ker} (\mathcal{P}) = \{1 \odot d | d \in \mathbb{R}^n\}$).

Thus, by setting the control gains according to the condition (16), the consensus protocol (6)-(8) enforces the $N$-port discrete-time passivity for the closed-loop system. If there is no delay, the gain setting condition (16) vanishes and in contrast to [11,12], we can choose arbitrary control gains, while enforcing passivity. On top of the passivity, with the condition (16), $f_i(k) = 0$, and extra damping $B_i^T$, the protocol (6)-(8) also achieves the position consensus among all agents. This is very similar to the results of continuous-time delayed
with some data-buffering for constant delays), as it allows useful for multiuser peer-to-peer haptics over the Internet to the discrete-time domain, for which the (open-loop) passivity of the MIs of Sec. II-B is indispensable (as that of the continuous-time robot is for the results of [15]).

The closed-loop passivity of our framework would be very useful for multiuser peer-to-peer haptics over the Internet (with some data-buffering for constant delays), as it allows any passive humans/devices to be connected while enforcing the closed-loop passivity; as well as the design of virtual world and device servo-loop to be separated from each other. See [16].

We perform a simulation for five 2-dim. MIs, with non-uniform spring matrices \( K_{ij} = k_{ij} I \) with \( k_{ij} \) in the range of 25 to 100 N/m; and with non-uniform constant delays in the range of 40 to 1000 ms. We set \( T_i(k) \) to be the same (10ms) for simplicity, although varying \( T_i(k) \) can be incorporated. See Fig. 2 for the simulation results. Due to the delay among the agents, it takes around 1 second to achieve position consensus both in X and Y directions. The consensus time can be improved by optimizing network topology, \( K_{ij} \) and \( B_i \), and it will be investigated in our future work. See [16] for a preliminary result in this direction.

IV. CONCLUSION AND FUTURE WORKS

We present a novel passivity-enforcing framework for multiple second-order mechanical integrators on undirected graph with constant non-uniform communication delay. By connecting multiple non-iterative passive MIs [16] through discrete-time springs over delayed links with some damping injection, the position consensus and the discrete-time closed-loop passivity are achieved. The control gains can be arbitrarily chosen regardless of the integration steps if there is no delay.

The proposed consensus framework is particularly promising for multiuser distributed haptic collaboration over the Internet as elucidated in [16]. There are also several possible directions for future research. In this paper, the indexes among all MIs are synchronized (i.e. same \( k \) shared). Yet, we believe this assumption can be dropped (i.e. \( k \) replaced by \( k_i \) for each MI). Extending our result to switching network topologies would be an interesting direction. Network topology optimization with limited network resource is also important for improving the performance. It is also interesting to extend our results here to general directed graphs.

REFERENCES