I. HOMEWORK 1

1. The pressure of a non-interacting gas (in units where $\hbar = c = k_B = 1$) is (this formulation is from Johns, Ellis, and Lattimer, 1996)

$$P(\mu, T) = \pm gT \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 \pm e^{-(E-\mu)/T} \right],$$  

(1)

where upper signs are for fermions and lower signs are for bosons, $g$ is the degeneracy factor, $k$ is the momentum, $m$ is the mass, and the “dispersion relation” is $E = \sqrt{k^2 + m^2}$. To convert to a one-dimensional integral, remember that

$$\frac{d^3k}{(2\pi)^3} \rightarrow \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{k^2 dk}{2\pi^2}$$  

(2)

(a) Use integration by parts to express the integrand in terms of the distribution function

$$f = \frac{1}{\exp [(E - \mu)/T] \pm 1}$$  

(3)

(b) For bosons, what is the restriction on $\mu$ as a function of $m$ to ensure that the distribution function is finite and non-negative for all $k$?

(c) Going back to the general case for both bosons and fermions, use a Taylor expansion of $E$ for small $k$ (and thus large $m$) to rewrite the distribution function in terms of a redefined chemical potential, $\tilde{\mu} = \mu - m$. This is the non-relativistic approximation.

(d) Now go back to the full expression for the pressure above, and take the case of bosons with $m = \mu = 0$ (i.e. photons). Compute the entropy density, $s$, and energy density, $\varepsilon$, from

$$s = \frac{\partial P}{\partial T} \quad \text{and} \quad \varepsilon = -P + Ts$$  

(4)

For photons, what is the correct value of $g$ and why?

(e) Finally, for fermions, use use the result from part (a) and the identity from Tooper (1969)

$$\int_0^\infty \frac{x^4 (x^2 + z^2)^{-1/2}}{1 + \exp (\sqrt{x^2 + z^2} - \phi)} \, dx = 3z^2 \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} c^n K_2(nz)$$  

(5)

to express the pressure as a sum over terms involving $K_2$, the modified Bessel function of the second kind. This will lead to the pressure in the “nondegenerate” or classical limit.

2. Complete exercise 1.5 on page 17 of Brown’s book.