Local analysis of co-dimension-one and co-dimension-two grazing bifurcations in impact microactuators

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Abstract

Impact microactuators rely on repeated collisions to generate gross displacements of a microelectromechanical machine element without the need for large applied forces. Their design and control rely on an understanding of the critical transition between non-impacting and impacting long-term system dynamics and the associated changes in system behavior. In this paper, we present three co-dimension-one, characteristically distinct transition scenarios associated with grazing conditions for a periodic response of an impact microactuator: a discontinuous jump to an impacting periodic response (associated with parameter hysteresis), a continuous transition to an impacting chaotic attractor, and a discontinuous jump to an impacting chaotic attractor. Using the concept of discontinuity mappings, a theoretical analysis is presented that predicts the character of each transition from a set of quantities that are computable in terms of system properties at grazing. Specifically, we show how this analysis can be applied to predict the bifurcation behavior on neighborhoods of two co-dimension-two bifurcation points that separate the co-dimension-one bifurcation scenarios. The predictions are validated against results from numerical simulations of a model impact microactuator.

Key words: Impact microactuators, grazing bifurcations, local analysis, discontinuity mappings

PACS:

1 Introduction

Precise displacement control and manipulation is required in microscopes, optical devices, and nanoscale data storage as well as during microsurgery. Adhesive forces, such as electrostatic forces, van der Waals forces, and surface
tension become significant when the characteristic length scale is less than a millimeter \([8,22]\). Therefore, manual assembly of microcomponents is difficult due to sticking, as this causes inaccuracy in positioning.

Micro actuators and micro systems have been extensively studied to obtain precise positioning \([1,2,4,5,12–17,19,22]\). Regular micro actuators used to produce measurable displacements would need large actuation forces and a long driving distance. In contrast, actuators based on impulsive forces provide a solution to this problem, as repeated impacts can be used to generate relatively large motion from the accumulation of many small displacements \([21]\). Recently, impact micro actuators have attracted a lot of attention due to ease of fabrication, capability of batch processing, robustness to environment, high accuracy, and high power \([5,15–17,19,22]\). For example, in the impact micro actuator studied by Mita et al \([17]\), controlled excitation is applied to the relative motion between a frame resting on a substrate and a movable block. For sufficiently large excitation, the relative motion results in repeated impacts between the block and the frame and resultant brief episodes of sliding motion of the frame along the substrate. Although each such sliding event contributes only on the order of hundredths of microns of displacement, periodic excitation may be used to generate controlled gross sliding.

Analysis and design of proper function of an impact micro actuator, such as that developed by Mita et al \([17]\) rely on an understanding of the possibly dramatic changes in system response that originate at the onset of impacting motions as a system parameter \(\mu\) (for example, the amplitude or frequency of the excitation) is varied. Specifically, we are concerned with bifurcations in the long-term system response that occur as \(\mu\) increases past some critical value \(\mu_{\text{grazing}}\), at which value there exists a periodic non-impacting oscillation of the movable block that achieves zero-relative-velocity contact with the frame. In a state-space description of the dynamics of the micro actuator, such zero-relative-velocity contact corresponds to a grazing contact between a state-space trajectory and a discontinuity surface, here representing the sudden changes in the velocities of the frame and the movable block that result from an impact. In contrast to periodic trajectories in smooth systems, the local description in the vicinity of a grazing trajectory is well-known to be non-differentiable with dramatic implications to the stability of the grazing trajectory and to its persistence under further parameter variations \([3,9,23]\). Indeed, as there is no advance warning of this instability, any local description must account for the nonsmooth character of the flow near the grazing trajectory.

The local dynamics in the vicinity of a grazing trajectory for \(\mu \approx \mu_{\text{grazing}}\) can be analyzed through the introduction of a discontinuity mapping that i) captures the local dynamics in the vicinity of the grazing contact including variations in time-of-flight to the discontinuity and the impact mapping; ii) can be entirely
characterized by conditions at the grazing contact; iii) is nonsmooth in the deviation from the point of grazing contact; and iv) can be studied to arbitrary order of accuracy [20]. Properly formulated, the discontinuity mapping thus introduces the correction to the otherwise smooth dynamics that is due to the brief interaction with the discontinuity [7,10,11,18,20].

In this paper, we revisit the dynamics of a piecewise smooth model of an experimental impact microactuator and analyze the transition between impacting and non-impacting system behavior using the discontinuity-mapping approach. In particular, we show how three previously identified transition scenarios correspond to co-dimension-one bifurcations and derive local mappings that accurately describe the dynamics in the vicinity of the grazing trajectory. Moreover, we identify the boundaries of the regions corresponding to the co-dimension-one bifurcation scenarios with co-dimension-two bifurcation points and apply the discontinuity-mapping approach to formulate approximate mappings that capture the local dynamics on open neighborhoods of these boundary points.

Specifically, Section 2 reviews the piecewise smooth formulation of the mathematical model of the impact microactuator and the bifurcation scenarios. In Section 3, we formulate the discontinuity-mapping methodology in the context of the impact microactuator. The local mappings derived in the vicinity of the two co-dimension-two bifurcation points are given in Section 4 together with numerical simulations showing the agreement between the local mapping and the results of numerical simulations of the piecewise smooth system. A concluding discussion is presented in Section 5.

2 A Model Impact Microactuator

2.1 Mathematical model

A schematic two-degree-of-freedom model of a micromachined impact actuator fabricated by Mita et al [17] is shown in Figure 1. The actual dimensions of the actuator are 3mm×3mm×0.6mm. Here, impulsive forces are generated as electrostatic actuation results in collisions between a silicon micromass and a stopper.

In the mechanical model of the Mita actuator shown above, the silicon micromass is represented by a movable block of mass $m_2$ that is connected to a frame of mass $m_1$ by a linear spring and damper. The stoppers and the electrode are rigidly fixed to the frame. The movable mass acts as one of the electrodes. The frame is assumed to rest on a horizontal substrate. Friction
between the frame and the ground is modelled using Coulomb friction during slip and Amonton’s law during stick. We denote the coefficient of static friction by $\mu_s$ and that of dynamic friction by $\mu_d$.

When a driving voltage $V(t)$ is applied between the electrodes, the movable block is accelerated toward the stopper until an impact occurs with the stopper. In the analysis below, we assume that the impact impulse is large enough to overcome the static friction between the frame and the ground for all impact velocities. As a result, an impact produces a small displacement of the frame. When a periodically varying voltage is applied, there are repeated impacts, thereby producing the needed displacement over some period of time.

The dynamics of the microactuator can be decomposed into distinct phases separated by the occurrence of impacts and the associated onset of slip as well as the subsequent cessation of slip through an instantaneous transition to stick. Specifically, we introduce the state vector

$$\mathbf{x} = \begin{pmatrix} q_1 & u_1 & q_2 & u_2 & \theta \end{pmatrix}^T,$$

where $q_1$ and $u_1$ are the displacement and velocity, respectively, of the frame relative to the ground; $q_2$ and $u_2$ are the displacement and velocity, respectively, of the movable block relative to the frame; and $\theta = \omega t$ is the phase of the sinusoidally varying driving voltage $V(t) = V_{\text{amp}} \cos(\omega t)$. 

Fig. 1. Schematic of the Mita et al [17] impact microactuator.
During stick, the equations of motion can then be written as

\[
\frac{dx}{dt} = f_{\text{stick}}(x) = \begin{pmatrix}
0 \\
0 \\
u_2 \\
\frac{1}{m_2} \left( \frac{\alpha V_{\text{amp}}^2 \cos^2 \theta}{(d - q_2)^2} - kq_2 - cu_2 \right) \\
\omega
\end{pmatrix}.
\]

(2)

Here, \( \alpha = \frac{1}{2} \epsilon_0 A \), where \( \epsilon_0 \) is the permittivity of free space, \( A \) is the overlap area, and \( d \) is the zero-voltage gap between the electrodes. These equations of motion are valid as long as

\[
h_{\text{front}}(x) = q_2 - \delta < 0,
\]

(3)

\[
h_{\text{back}}(x) = q_2 + \delta > 0,
\]

(4)

\[
h_{\text{stick}+}(x) = kq_2 + cu_2 - \frac{\alpha V_{\text{amp}}^2 \cos^2 \theta}{(d - q_2)^2} - \mu_s N < 0,
\]

(5)

and

\[
h_{\text{stick}−}(x) = kq_2 + cu_2 - \frac{\alpha V_{\text{amp}}^2 \cos^2 \theta}{(d - q_2)^2} + \mu_s N > 0,
\]

(6)

where \( N \) is the normal reaction from the ground. We assume that gravity is the only external force, in which case \( N = (m_1 + m_2)g \), where \( g \approx 9.81 \text{ m/s}^2 \) is the acceleration of gravity.

During slip, the equations of motion can be written as

\[
\frac{dx}{dt} = f_{\text{slip}±}(x) = \begin{pmatrix}
u_1 \\
\frac{1}{m_1} F \\
u_2 \\
\frac{1}{m_2} \left( \frac{\alpha V_{\text{amp}}^2 \cos^2 \theta}{(d - q_2)^2} - kq_2 - cu_2 \right) - \frac{1}{m_1} F \\
\omega
\end{pmatrix},
\]

(7)

where \( F = \left( kq_2 + cu_2 - \frac{\alpha V_{\text{amp}}^2 \cos^2 \theta}{(d - q_2)^2} \pm \mu_d N \right) \) and we use the upper sign when \( u_1 > 0 \) and the lower sign when \( u_1 < 0 \). Again, the corresponding equations of motion are valid as long as

\[
h_{\text{front}}(x) = q_2 - \delta < 0,
\]

(8)

\[
h_{\text{back}}(x) = q_2 + \delta > 0,
\]

(9)

and

\[
h_{\text{slip}}(x) = u_1 \neq 0.
\]

(10)
At the moment that contact is established between the movable block and the stopper, $h_{\text{front}}(x)$ or $h_{\text{back}}(x)$ equal zero. Assuming an inelastic collision with a coefficient of restitution $e$ and using conservation of momentum, the function that maps the state immediately prior to impact to the state immediately after impact is given by the jump map

$$g_{\text{impact}}(x) = \left( q_1 u_1 + \frac{(1+e)m_2}{m_1+m_2} u_2 q_2 - eu_2 \theta \right)^T.$$  \hspace{1cm} (11)

The transition from slip to stick occurs as the velocity of the frame relative to the ground becomes zero, i.e., as $h_{\text{slip}}(x)$ equals zero. Although there is a discontinuous change in the vector field as a result of this transition, there is no associated instantaneous change of state, i.e.,

$$g_{\text{stick}}(x) = x.$$  \hspace{1cm} (12)

In the numerical results reported below, we have normalized mass, length, time, and voltage by $m_2$, $d$, $\sqrt{m_2/k}$ and $V_0$, respectively, where $V_0$ is some characteristic voltage. The nondimensional system parameter values used for numerical computations are $m_1 = 5$, $m_2 = 1$, $k = 1$, $c = 0.04$, $d = 1$, $d = 0.5$, $e = 0.8$, $\mu_s = 0.4$, $\mu_d = 0.27$, and $\alpha = 1$. As shown in the authors’ previous paper [24], these parameters are chosen to fit the experimental data reported by Mita et al [17].

2.2 Transition Scenarios

The piecewise smooth dynamical system formulated above possesses two types of recurrent solutions, namely, those with impacts (and thus sliding episodes) and those without impacts (and no sliding episodes). In a previous paper (Zhao et al [24]), we have numerically investigated representative transitions between impacting and non-impacting recurrent solutions that occur under changes in system parameters. Here, we review the different transition scenarios found previously as well as describe additional scenarios that occur under parameter variations through isolated points in the two-dimensional parameter space given by $V_{\text{amp}}$ and $\omega$.

In the absence of forcing, the only recurrent motion available to the impact microactuator is the trivial equilibrium state. For sufficiently small, but nonzero, values of $V_{\text{amp}}$ the movable mass exhibits a periodic oscillation of the same period as that of the forcing term (note that the voltage input appears squared in the forcing term, i.e., the angular frequency of the forcing term equals $2\omega$) but without impacting the front or back stopper. Indeed, as the forcing term is always in the positive direction, the oscillatory motion of the movable mass
is shifted in the direction of positive values of \(q_2\). It follows that, under increasing values of \(V_{\text{amp}}\), impacting motions that impact with \(q_2 = \delta\) will occur before impacting motions that impact with \(q_2 = -\delta\). In this and the previous paper, we therefore focus on recurrent motions that only impact with \(q_2 = \delta\) and, consequently, for which the relative velocity between the frame and the substrate is positive during sliding episodes.

For notational convenience, consider the following notation

\[
\mathcal{D} = \{ \mathbf{x} \mid h^D (\mathbf{x}) \overset{\text{def}}{=} h_{\text{front}} (\mathbf{x}) = 0 \} \tag{13}
\]

\[
\mathcal{D}^+ = \{ \mathbf{x} \in \mathcal{D} \mid f_{\text{stick}} (\mathbf{x}) \bullet h^D_X (\mathbf{x}) = u_2 > 0 \} \tag{14}
\]

\[
\mathcal{D}^0 = \{ \mathbf{x} \in \mathcal{D} \mid f_{\text{stick}} (\mathbf{x}) \bullet h^D_X (\mathbf{x}) = u_2 = 0 \} \tag{15}
\]

\[
\mathcal{D}^- = \{ \mathbf{x} \in \mathcal{D} \mid f_{\text{stick}} (\mathbf{x}) \bullet h^D_X (\mathbf{x}) = u_2 < 0 \} \tag{16}
\]

As per the previous discussion, for sufficiently small values of \(V_{\text{amp}}\), the recurrent motion is contained within the region \(h^D (\mathbf{x}) < 0\). Since \(g_{\text{impact}}\) maps \(\mathcal{D}^+\) to \(\mathcal{D}^-\), trajectories that reach \(\mathcal{D}^+\) experience an instantaneous jump to \(\mathcal{D}^-\) (as the incoming velocity \(u_2 > 0\) is changed to an outgoing velocity \(-eu_2 < 0\)).

By a grazing periodic trajectory, we refer to a periodic trajectory on which there exists a locally unique point \(\mathbf{x}^* \in \mathcal{D}^0\), since such a trajectory experiences no jump in state upon reaching \(\mathcal{D}\). As discussed in Zhao et al [24], we can use a Newton method (see appendix) to numerically locate parameter values for \(\omega\) and \(V_{\text{amp}}\) for which grazing periodic trajectories exist. Indeed, over the region of interest in this paper (\(0.46 \leq \omega \leq 0.51\)), we find a curve \(\Gamma = \{ (\omega, V_{\text{amp}}) \mid V_{\text{amp}} = V^*_{\text{amp}} (\omega) \}\) in parameter space, such that there exists a grazing periodic trajectory of the fundamental period of the forcing term for every choice of parameters on \(\Gamma\) (see Figure 4, below).

We now consider recurrent dynamics of the impact microactuator on some neighborhood in parameter space of arbitrary points on the grazing curve \(\Gamma\). To visualize the results of the numerical study, we introduce a Poincaré section \(\mathcal{P}\) corresponding to the zero-level surface of the event function \(h^P (\mathbf{x}) = f_{\text{stick}} (\mathbf{x}) \bullet h^D_X (\mathbf{x}) = u_2\) for \(u_2\) decreasing. (Note that \(\mathcal{P} \cap \mathcal{D}\) is contained in \(\mathcal{D}^0\).) In the absence of impacts, points on \(\mathcal{P}\) correspond to local maxima in the value of \(q_2\) along system trajectories (since \(\dot{q}_2 = u_2\)). In the presence of impacts, trajectories that reach \(\mathcal{D}^+\) jump across \(\mathcal{P}\) without intersecting \(\mathcal{P}\). We represent such crossings by the virtual point of intersection with \(\mathcal{P}\) of the corresponding forward trajectory segment in the absence of the jump in velocity, as suggested in Figure 2.

Figures 3a)-c) show three distinct bifurcation scenarios corresponding to transversal one-parameter variations across some selected points on \(\Gamma\). Further such numerical studies establish the existence of three distinct regions on \(\Gamma\), referred to here as \(\Gamma_I\), \(\Gamma_{II}\), and \(\Gamma_{III}\), for which the bifurcation scenarios in the
Fig. 2. Intersections of nonimpacting and impacting trajectories with the Poincaré section \( \mathcal{P} \). The nonimpacting trajectory reaches its local maximum in \( q_2 \) at the intersection. The impacting trajectory reaches \( \mathcal{P} \) virtually by neglecting the existence of the discontinuity surface \( \mathcal{D} \).

Different panels of Figure 3 are representative.

As shown in Figure 3a), variations in \( V_{\text{amp}} \) across \( \Gamma_I \) are associated with a discontinuous transition of the asymptotic motion from a nonimpacting to an impacting periodic trajectory as \( V_{\text{amp}} \) is increased above the grazing parameter value \( V_{\text{amp}}^* (\omega) \); and from an impacting to a nonimpacting periodic trajectory as \( V_{\text{amp}} \) is decreased below \( V_{\text{amp}}^{\text{sn1}} (\omega) < V_{\text{amp}}^* (\omega) \) corresponding to a cyclic fold bifurcation. For \( V_{\text{amp}} \in (V_{\text{amp}}^{\text{sn1}}, V_{\text{amp}}^*) \), three distinct periodic trajectories (one nonimpacting and two impacting) coexist. Of these, the nonimpacting trajectory and the larger-impact-velocity impacting trajectory are stable, while the lower-impact-velocity impacting trajectory is unstable. The coexistence of multiple attractors implies the possibility of parameter hysteresis in the long-term response of the impact actuator.

Variations in \( V_{\text{amp}} \) across \( \Gamma_{II} \) are associated with a continuous transition of the asymptotic dynamics from a nonimpacting periodic trajectory to an impacting chaotic attractor and back as \( V_{\text{amp}} \) is increased above and decreased below \( V_{\text{amp}}^* (\omega) \). In contrast to the transition across \( \Gamma_I \), the nonimpacting trajectory persists beyond the grazing contact (albeit with impacts and sliding episodes), but experiences a discontinuous change in stability as one of its eigenvalues jumps to \(-\infty\). Under further increases in \( V_{\text{amp}} \), this unstable impacting periodic trajectory becomes stable again in a period-doubling bifurcation at \( V_{\text{amp}}^{pd} (\omega) \). Moreover, as suggested in the magnified portion of Figure 3 d), the impacting chaotic attractor undergoes an inverse period-doubling cascade to an impacting period-two trajectory that connects to the branch of period-two trajectories born at the period-doubling bifurcation through a grazing bifurcation.
Fig. 3. Schematic bifurcation scenarios associated with the switching between impacting motions (black) and non-impacting motions (gray) (G=grazing contact, SN=saddle-node bifurcation, PD=period-doubling bifurcation, C=global crisis). Here, solid curves correspond to stable periodic motions and dashed curves to unstable periodic motions. The black regions correspond to impacting chaotic attractors. Panel a), b), and c) describe the transitions under variations in $V_{\text{amp}}$ across $\Gamma_I$, $\Gamma_{II}$, and $\Gamma_{III}$, respectively. Panel d) and e) show enlargement of the corresponding areas in b) and c), respectively.

The continuous nature of the transitions between system attractors in the case of variations across $\Gamma_{II}$ implies that parameter hysteresis cannot occur.

Finally, variations in $V_{\text{amp}}$ across $\Gamma_{II}$ are associated with a discontinuous transition of the asymptotic dynamics from a nonimpacting periodic trajectory to an impacting chaotic attractor as $V_{\text{amp}}$ is increased above $V_{\text{amp}}^{sn_1}(\omega) > V_{\text{amp}}^{*}(\omega)$; and from an impacting chaotic attractor to a nonimpacting periodic trajectory as $V_{\text{amp}}$ is decreased below $V_{\text{amp}}^{cr}(\omega) < V_{\text{amp}}^{sn_2}(\omega)$, where $V_{\text{amp}}^{*}(\omega) < V_{\text{amp}}^{cr}(\omega)$, corresponding to the disappearance of the basin of attraction of the chaotic attractor in a global crisis bifurcation. For $V_{\text{amp}} \in \left(V_{\text{amp}}^{*}(\omega), V_{\text{amp}}^{sn_2}(\omega)\right)$, three distinct periodic trajectories (two nonimpacting and one impacting) coexist. Of these, the unstable impacting trajectory originates in the grazing contact and persists as $V_{\text{amp}}$ is increased beyond $V_{\text{amp}}^{sn_2}(\omega)$. It eventually becomes stable in a period-doubling bifurcation at $V_{\text{amp}}^{pd}(\omega)$. Of the nonimpacting trajectories, the small-amplitude motion is stable, while the large-amplitude motion (that experiences grazing contact at $V_{\text{amp}}^{*}(\omega)$ and that collides with the chaotic attractor at $V_{\text{amp}}^{cr}(\omega)$) is unstable. Again, the coexistence of multiple attractors implies the possibility of parameter hysteresis in the long-term response of the impact actuator.

From these scenarios, we find it useful to collect in a diagram of parameter
space, the grazing curve $\Gamma$, the locus of the saddle-node bifurcations that occur when crossing $\Gamma_I$, the locus of the period-doubling bifurcations that occur when crossing $\Gamma_{II}$ and $\Gamma_{III}$, the locus of the saddle-node bifurcations that occur when crossing $\Gamma_{III}$, and the locus of the global crisis of the chaotic attractor that occurs when crossing $\Gamma_{III}$. As can be seen from Figure 4, the saddle-node bifurcation curve associated with $\Gamma_I$ and the period-doubling bifurcation curve appear to terminate at the boundary between $\Gamma_I$ and $\Gamma_{II}$ and to be tangent to $\Gamma$ at this point. Similarly, the crisis and saddle-node bifurcation curves associated with $\Gamma_{III}$ terminate at the boundary between $\Gamma_{II}$ and $\Gamma_{III}$ and are tangent to $\Gamma$ at that point.

Additional transition scenarios may be found by considering transversal one-parameter variations across the boundary points between $\Gamma_I$ and $\Gamma_{II}$ and between $\Gamma_{II}$ and $\Gamma_{III}$, respectively, see Figure 5a)-b). As seen in Figure 5a) and in contrast to the transitions across $\Gamma_I$ or $\Gamma_{II}$, crossing the boundary point between $\Gamma_I$ and $\Gamma_{II}$ is associated with a continuous transition of the asymptotic dynamics from a nonimpacting periodic trajectory to an impacting periodic trajectory and back as $V_{\text{amp}}$ is increased above and decreased below $V^*_{\text{amp}}(\omega)$. The transition in Figure 5b), on the other hand, shows significant similarities with the transition across $\Gamma_{II}$, although here the grazing trajectory is nonhyperbolic. As the additional transition scenarios occur only near isolated points in the $\omega - V_{\text{amp}}$ parameter space, we refer to the corresponding grazing bifurcations as co-dimension two, thereby distinguishing them from the co-dimension-one bifurcations associated with passage through $\Gamma_I$, $\Gamma_{II}$, or $\Gamma_{III}$. 

**Fig. 4.** Collection of bifurcation curves in the parameter space, where $\Gamma$ represents the grazing curve; $V^{sn1}_{\text{amp}}$ represents the locus of saddle-node bifurcations when crossing $\Gamma_I$; $V^{pd}_{\text{amp}}$ represents the locus of period-doubling bifurcations when crossing $\Gamma_{II}$ and $\Gamma_{III}$; $V^{sn2}_{\text{amp}}$ represents the locus of saddle-node bifurcations when crossing $\Gamma_{III}$; $V_{\text{cramp}}$ represents the locus of the global crisis of the chaotic attractor when crossing $\Gamma_{III}$. Panels b) and c) show enlargements of neighborhoods of the boundary point between $\Gamma_I$ and $\Gamma_{II}$ and the boundary point between $\Gamma_{II}$ and $\Gamma_{III}$, respectively.
a) $\omega = 0.4878$  

b) $\omega = 0.4803$

Fig. 5. Schematic bifurcation scenarios associated with the switching between impacting motions (black) and non-impacting motions (gray), at the boundary between $\Gamma_I$ and $\Gamma_{II}$ in Panel a) and those at the boundary between $\Gamma_{II}$ and $\Gamma_{III}$ in Panel b) (G=grazing contact, PD=period-doubling bifurcation). Here, solid curves correspond to stable periodic motions and dashed curves to unstable periodic motions. The black regions correspond to impacting chaotic attractors.

We note that bifurcation scenarios similar to those observed here in the case of $\Gamma_I$ and $\Gamma_{II}$ have previously been documented in piecewise smooth dynamical systems, for example, in linear impact oscillators (Foale & Bishop [9]) and in friction oscillators (Dankowicz & Nordmark [6]). The transition scenario associated with $\Gamma_{III}$ is inherent in the nonlinearity of the electrostatic forces and is a consequence of a dynamic pull-in phenomenon in electrostatically attracting microelectrodes.

The goal of the following sections is to develop closed-form maps that approximate the dynamics near the grazing trajectory in state and parameter space and that allow accurate predictions of the long-term response without the need to simulate the original piecewise smooth system of differential equations. Indeed, as simulations of recurrent motions that involve many near-zero-velocity impacts require very high numerical accuracy and long integration times, such closed-form maps can be enormously beneficial.

3 Discontinuity mappings

We wish to associate a Poincaré mapping $\mathbf{P}$ with the Poincaré section $\mathcal{P}$ introduced above.

Ignore, for a moment, the jump map associated with the discontinuity $\mathcal{D}$ and assume that the dynamics are governed entirely by the vector field $\mathbf{f}_{\text{stick}}$ and constrained to a constant-$q_1$ slice of the submanifold $\mathcal{S}$ corresponding to the zero-level surface of the event function $h^\mathcal{S}(\mathbf{x}) = u_1$. Suppose that for some values $\omega = \omega^{\text{ref}}$ and $V_{\text{amp}} = V_{\text{amp}}^{\text{ref}}$, the forward trajectory based at a point
$x_{\text{ref}} \in \mathcal{P}$ intersects $\mathcal{P}$ transversally after some time $t_{\text{ref}}$, i.e., that
\begin{equation}
  h^\mathcal{P} \left( \Phi_{\text{stick}} \left( x_{\text{ref}}, t_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) \right) = 0
\end{equation}
and
\begin{equation}
  h^\mathcal{P}_{i} \left( \Phi_{\text{stick}} \left( x_{\text{ref}}, t_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) \right) \cdot f_{\text{stick}} \left( \Phi_{\text{stick}} \left( x_{\text{ref}}, t_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) \right) < 0.
\end{equation}
where $\Phi_{\text{stick}}$ is the smooth flow corresponding to the vector field $f_{\text{stick}}$. Here, the latter condition corresponds to the requirement that $u_2$ be decreasing, i.e., that the acceleration of the movable mass relative to the frame be negative.

Now introduce the function
\begin{equation}
  F (x, t, \omega, V_{\text{amp}}) = h^\mathcal{P} \left( \Phi_{\text{stick}} (x, t; \omega, V_{\text{amp}}) \right).
\end{equation}
It follows that
\begin{equation}
  F \left( \mathbf{x}_{\text{ref}}, t_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) = 0
\end{equation}
and
\begin{equation}
  F_t \left( \mathbf{x}_{\text{ref}}, t_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) < 0.
\end{equation}
The implicit function theorem implies that there exists a unique smooth function $\tau (x, \omega, V_{\text{amp}})$ defined on a neighborhood of $(\mathbf{x}_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}})$, such that
\begin{equation}
  \tau \left( \mathbf{x}_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}} \right) = t_{\text{ref}}
\end{equation}
and
\begin{equation}
  F (x, \tau (x, \omega, V_{\text{amp}}), \omega, V_{\text{amp}}) \equiv 0,
\end{equation}
i.e., $\tau (x, \omega, V_{\text{amp}})$ is the time of flight from $x$ back to $\mathcal{P}$. A smooth Poincaré mapping $\mathcal{P}$ can now be defined on a neighborhood of $(\mathbf{x}_{\text{ref}}, \omega_{\text{ref}}, V_{\text{amp}})$ by the expression
\begin{equation}
  \mathcal{P} (x, \omega, V_{\text{amp}}) = \Phi_{\text{stick}} (x, \tau (x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}).
\end{equation}

If we reintroduce the nontrivial jump map $g_{\text{impact}}$ associated with $\mathcal{D}$, the above expression is still valid as long as $h^\mathcal{D} (x) \leq 0$. If, instead, $h^\mathcal{D} (x) > 0$, we recognize that the point $x$ corresponds to a virtual point of intersection as per the definition in the previous section that cannot actually be reached by the piecewise smooth dynamical system. It would, nevertheless, be convenient if we could again define the Poincaré mapping $\mathcal{P}$ by the above formula, but possibly including an initial correction to account for the virtual nature of the initial point $x$. To this end, consider the Poincaré mapping $\mathcal{P}$ defined by
\begin{equation}
  \mathcal{P} (x, \omega, V_{\text{amp}}) = \Phi_{\text{stick}} (\mathcal{D} (x, \omega, V_{\text{amp}}), \tau (\mathcal{D} (x, \omega, V_{\text{amp}}), \omega, V_{\text{amp}}); \omega, V_{\text{amp}}),
\end{equation}
where the *discontinuity mapping* $D$ maps $x$ to some point on $P$ in such a way that the subsequent dynamics respect those of the corresponding actual trajectory (For more discussion of the concept of discontinuity mappings and their derivation, see also [6,7,10,18,20]).

To arrive at an expression for $D$ consider the trajectory segments shown in Figure 6. Here, an incoming trajectory (solid) in $S$ governed by the vector field $f_{\text{stick}}$ reaches the discontinuity surface $D$ at a point $x_{\text{in}} \in D^+$, experiences a jump to a point $g_{\text{impact}}(x_{\text{in}}) \in D^-$, flows under the vector field $f_{\text{slip}^+}$ until reaching the stick manifold $S$ at a point $x_{\text{out}}$, and then continues to flow in $S$ under the vector field $f_{\text{stick}}$. The dashed trajectory segments correspond to a flow in $S$ governed by the vector field $f_{\text{stick}}$ from $x_{\text{in}}$ forward in time until reaching $P$ at a point $x_0$; and from $x_{\text{out}}$ backward in time until reaching $P$ at a point $x_1$. The sought correction to the smooth flow given by $\Phi_{\text{stick}}$ is then obtained by mapping $x_0$ to $x_1$, as this correctly accounts for the effects of the jump map and the subsequent sliding episode.

![Fig. 6. Trajectories associated with the discontinuity mapping D. Here S stands for the stick manifold, D is the discontinuity surface, and P is the Poincaré section. We note that S, D, and P are all 4 dimensional hyper surfaces in a 5 dimensional space.](image)

Thus, given an initial point $x$ on $P$, such that $h^D(x) > 0$, we define $D$ as the composition of the following steps:

1. Flow for a time $t_1 < 0$ with the vector field $f_{\text{stick}}$ until reaching $D$;
2. Apply the jump map $g_{\text{impact}}$;
3. Flow for a time $t_2 > 0$ with the vector field $f_{\text{slip}^+}$ until reaching $S$;
4. Flow for a time $t_3 < 0$ with the vector field $f_{\text{stick}}$ until reaching $P$.

To arrive at a functional expression for $D$, we seek to express the flow times in terms of the corresponding initial conditions in state space and the corresponding parameter values.
Suppose, in particular, that for \( \omega = \omega^* \) and \( V_{\text{amp}} = V^*_{\text{amp}} \), there exists a point \( x^* \), such that

\[
\begin{align*}
    h^D(x^*) &= 0, \quad (26) \\
    h^P(x^*) &= h^D_i(x^*) \cdot f^i_{\text{stick}}(x^*, \omega^*, V^*_{\text{amp}}) = 0, \quad (27) \\
    h^S(x^*) &= 0, \quad (28) \\
    h^S_i(x^*) \cdot f^i_{\text{slip}+}(x^*, \omega^*, V^*_{\text{amp}}) &< 0, \quad (29)
\end{align*}
\]

and

\[
    a^*(\omega^*, V^*_{\text{amp}}) \overset{\text{def}}{=} h^D_j(x^*) \cdot f^j_{\text{stick}}(x^*, \omega^*, V^*_{\text{amp}}) = \left( h^D_{ij}(x^*) \cdot f^i_{\text{stick}}(x^*, \omega^*, V^*_{\text{amp}}) + h^D_i(x^*) \cdot f^i_{\text{stick}, j}(x^*, \omega^*, V^*_{\text{amp}}) \right) \cdot f^j_{\text{stick}}(x^*, \omega^*, V^*_{\text{amp}}) < 0.
\]

Clearly, \( x^* \in \mathcal{P} \cap \mathcal{S} \) is a point of simple grazing contact with \( \mathcal{D} \) corresponding to a local maximum in \( h^D \) along a trajectory segment of the vector field \( f_{\text{stick}} \) based at \( x^* \).

**Step 1.** Suppose that \( h^D(x) > 0 \) and consider the function

\[
    E^{(1)}(x, y, t, \omega, V_{\text{amp}}) = t \sqrt{\frac{h^D(x) - h^D(\Phi_{\text{stick}}(x, -t; \omega, V_{\text{amp}})) - th^P(x)}{t^2}} - y.
\]

Since

\[
    \frac{h^D(x) - h^D(\Phi_{\text{stick}}(x, -t; \omega, V_{\text{amp}})) - th^P(x)}{t^2}
\]

is bounded in \( t \) for \( t \approx 0 \) (as seen by Taylor expanding the numerator), it follows that

\[
    E^{(1)}(x^*, 0, 0, \omega^*, V^*_{\text{amp}}) = 0
\]

and

\[
    E^{(1)}_x(x^*, 0, 0, \omega^*, V^*_{\text{amp}}) = \sqrt{-\frac{a^*(\omega^*, V^*_{\text{amp}})}{2}}.
\]

The implicit function theorem now implies the existence of a unique smooth function \( \tau^{(1)}(x, y, \omega, V_{\text{amp}}) \) on an open neighborhood of \( (x^*, 0, \omega^*, V^*_{\text{amp}}) \), such that

\[
    \tau^{(1)}(x^*, 0, \omega^*, V^*_{\text{amp}}) = 0
\]

and

\[
    E^{(1)}(x, y, \tau^{(1)}(x, y, \omega, V_{\text{amp}}), \omega, V_{\text{amp}}) \equiv 0.
\]

It follows from the definition of \( E^{(1)} \) that \( t_1 = -\tau^{(1)}(x, \sqrt{h^D(x)}, \omega, V_{\text{amp}}) \).

As the smooth function \( \tau^{(1)}(x, y, \omega, V_{\text{amp}}) \) is implicitly defined by Eq. (35),
we can compute arbitrary partial derivatives of $\tau^{(1)}$ at $(x^*, 0, \omega^*, V_{\text{amp}}^*)$ using implicit differentiation and demanding that all partial derivatives of $E^{(1)}$ must vanish at $(x^*, 0, \omega^*, V_{\text{amp}}^*)$. Now let

$$D_1 (x, y, \omega, V_{\text{amp}}) = \Phi_{\text{stick}} (x, -\tau^{(1)} (x, y, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}).$$  \hspace{1cm} (36)$$

Since $\Phi_{\text{stick}}$ is smooth, we can compute arbitrary partial derivatives of $D_1$ at $(x^*, 0, \omega^*, V_{\text{amp}}^*)$. Step 1 is completed by expanding $D_1 (x, y, \omega, V_{\text{amp}})$ to desired order in the deviation from $(x^*, 0, \omega^*, V_{\text{amp}}^*)$ and substituting $y = \sqrt{h^D (x)}$.

**Step 2.** Since $g_{\text{impact}}$ is smooth, it can be expanded to desired order in the deviation from $(x^*, \omega^*, V_{\text{amp}}^*)$, after which the result of step 1 may be substituted for $x$ thus completing step 2.

**Step 3.** Now consider the function

$$E^{(2)} (x, t, \omega, V_{\text{amp}}) = h^S (\Phi_{\text{slip}^+} (x, t; \omega, V_{\text{amp}})), \hspace{1cm} (37)$$

where $\Phi_{\text{slip}^+}$ is the flow corresponding to the vector field $f_{\text{slip}^+}$. Then,

$$E^{(2)} (x^*, 0, \omega^*, V_{\text{amp}}^*) = h^S (x^*) = 0 \hspace{1cm} (38)$$

and

$$E^{(2)}_t (x^*, 0, \omega^*, V_{\text{amp}}^*) = h^S (x^*) \cdot f_{\text{slip}^+} (x^*, \omega^*, V_{\text{amp}}^*) < 0. \hspace{1cm} (39)$$

The implicit function theorem again implies the existence of a unique smooth function $\tau^{(2)} (x, \omega, V_{\text{amp}})$ on an open neighborhood of $(x^*, \omega^*, V_{\text{amp}}^*)$, such that

$$\tau^{(2)} (x^*, \omega^*, V_{\text{amp}}^*) = 0 \hspace{1cm} (40)$$

and

$$E^{(2)} (x, \tau^{(2)} (x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}) \equiv 0. \hspace{1cm} (41)$$

It follows from the definition of $E^{(2)}$ that $t_2 = \tau^{(2)} (x, \omega, V_{\text{amp}})$.

As the smooth function $\tau^{(2)} (x, \omega, V_{\text{amp}})$ is implicitly defined by Eq. (41), we can compute arbitrary partial derivatives of $\tau^{(2)}$ at $(x^*, \omega^*, V_{\text{amp}}^*)$ using implicit differentiation and demanding that all partial derivatives of $E^{(2)}$ must vanish at $(x^*, \omega^*, V_{\text{amp}}^*)$. Now let

$$D_2 (x, \omega, V_{\text{amp}}) = \Phi_{\text{slip}^+} (x, \tau^{(2)} (x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}). \hspace{1cm} (42)$$

Since $\Phi_{\text{slip}^+}$ is smooth, we can compute arbitrary partial derivatives of $D_2$ at $(x^*, \omega^*, V_{\text{amp}}^*)$. Step 3 is completed by expanding $D_2 (x, \omega, V_{\text{amp}})$ to desired order in the deviation from $(x^*, \omega^*, V_{\text{amp}}^*)$ and substituting the result of step 2 for $x$. 

15
Step 4. Finally, consider the function

$$E^{(3)}(x, t, \omega, V_{\text{amp}}) = h^P (\Phi_{\text{stick}}(x, -t, \omega, V_{\text{amp}})), \quad (43)$$

Then,

$$E^{(3)}(x^*, 0, \omega^*, V_{\text{amp}}^*) = h^P (x^*) = 0 \quad (44)$$

and

$$E_t^{(3)}(x^*, 0, \omega^*, V_{\text{amp}}^*) = -a^*(\omega^*, V_{\text{amp}}^*) > 0. \quad (45)$$

The implicit function theorem again implies the existence of a unique smooth function $\tau^{(3)}(x, \omega, V_{\text{amp}})$ on an open neighborhood of $(x^*, \omega^*, V_{\text{amp}}^*)$, such that

$$\tau^{(3)}(x^*, \omega^*, V_{\text{amp}}^*) = 0 \quad (46)$$

and

$$E^{(3)}(x, \tau^{(3)}(x, \omega, V_{\text{amp}}), \omega, V_{\text{amp}}) \equiv 0. \quad (47)$$

It follows from the definition of $E^{(3)}$ that $t_3 = -\tau^{(3)}(x, \omega, V_{\text{amp}})$.

As the smooth function $\tau^{(3)}(x, \omega, V_{\text{amp}})$ is implicitly defined by Eq. (47), we can compute arbitrary partial derivatives of $\tau^{(3)}$ at $(x^*, \omega^*, V_{\text{amp}}^*)$ using implicit differentiation and demanding that all partial derivatives of $E^{(3)}$ must vanish at $(x^*, \omega^*, V_{\text{amp}}^*)$. Now let

$$D_3(x, \omega, V_{\text{amp}}) = \Phi_{\text{stick}}(x, -\tau^{(3)}(x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}) \quad (48)$$

Since $\Phi_{\text{stick}}$ is smooth, we can compute arbitrary partial derivatives of $D_3$ at $(x^*, \omega^*, V_{\text{amp}}^*)$. Step 4 is completed by expanding $D_3(x, \omega, V_{\text{amp}})$ to desired order in the deviation from $(x^*, \omega^*, V_{\text{amp}}^*)$ and substituting the result of step 3 for $x$.

Now consider the compositions

$$\tilde{D}(x, y, \omega, V_{\text{amp}}) =$$

$$= \begin{cases} 
\text{Id} & \text{when } h^D(x) \leq 0 \\
D_3(D_2(g_{\text{impact}}(D_1(x, y, \omega, V_{\text{amp}})); \omega, V_{\text{amp}})); \omega, V_{\text{amp}}) & \text{when } h^D(x) > 0 
\end{cases}$$

and

$$\tilde{P}(x, \omega, V_{\text{amp}}) = \Phi_{\text{stick}}(x, \tau(x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}). \quad (49)$$

It follows that

$$D(x, \omega, V_{\text{amp}}) = \tilde{D}(x, \sqrt{h^D(x)}, \omega, V_{\text{amp}}) \quad (50)$$

and

$$P(x, \omega, V_{\text{amp}}) = \tilde{P}(D(x, \omega, V_{\text{amp}}), \omega, V_{\text{amp}}). \quad (51)$$
We note that although \( \tilde{D} \) and \( \tilde{P} \) are smooth on an open neighborhood of \( (x^*, 0, \omega^*, V_{\text{amp}}^*) \) and \( (x^*, \omega^*, V_{\text{amp}}^*) \), respectively, it is not true that \( D \) and \( P \) are smooth on an open neighborhood of \( (x^*, \omega^*, V_{\text{amp}}^*) \) due to the substitution \( y = \sqrt{h^D(x)} \).

While \( \tilde{P} \) results in no change in \( q_1 \), the inclusion of the jump mapping \( g_{\text{impact}} \) and the associated sliding episode yields a discrete change in \( q_1 \) contained within the discontinuity mapping \( D \) and, consequently, in the composite Poincaré mapping \( P \). As the system is invariant under variations in \( q_1 \) and as, for low-velocity impacts \( P(x) \in \mathcal{P} \cap \mathcal{S} \), we consider the reduced system obtained by restricting attention to the third and fifth components of \( P \).

4 Numerical results

We now apply the discontinuity-mapping approach to yield explicit numerical expressions for the composite Poincaré mapping \( P(x, \omega, V_{\text{amp}}) \) on neighborhoods of selected grazing periodic trajectories as found numerically using the Newton approach outlined in the appendix and described previously. For convenience, we let \( x = x^* + \Delta x, \omega = \omega^* + \Delta \omega, \) and \( V_{\text{amp}} = V_{\text{amp}}^* + \Delta V_{\text{amp}} \), and use \( (\Delta x, \Delta \omega, \Delta V_{\text{amp}}) \) as local coordinates.

The grazing periodic trajectory at the co-dimension-two point at the boundary between \( \Gamma_I \) and \( \Gamma_{II} \) corresponds to

\[
\omega^* \approx 0.487809, \ V_{\text{amp}}^* \approx 0.171067 \text{ and } x^* \approx \begin{pmatrix} \cdot & 0 & 0.5 & 5.496544 \end{pmatrix}^T.
\]

Following the methodology in the previous section and the comments made
in the last paragraph of the previous section, we find

\[
\tilde{P}(x, \omega, V_{\text{amp}}) - x^* = \\
\begin{bmatrix}
0.936621\Delta q_2 + 0.775788 \Delta V_{\text{amp}} - 0.174538 \Delta \omega \\
+ (0.111825 \Delta q_2 - 0.0103004 \Delta \theta + 0.960382 \Delta V_{\text{amp}} - 0.627897 \Delta \omega) \Delta q_2 \\
- (0.103166 \Delta \theta + 0.14153 \Delta V_{\text{amp}} + 0.669646 \Delta \omega) \Delta \theta \\
+ 3.45355 \Delta V_{\text{amp}}^2 - 1.52077 \Delta \omega^2 - 3.21475 \Delta V_{\text{amp}} \Delta \omega \\
0.142633 \Delta q_2 + 0.825198 \Delta \theta + 0.98176 \Delta V_{\text{amp}} + 5.82623 \Delta \omega \\
+ (0.394304 \Delta q_2 - 0.148787 \Delta \theta + 2.22656 \Delta V_{\text{amp}} - 0.441188 \Delta \omega) \Delta q_2 \\
+ (0.0138705 \Delta \theta - 2.25913 \Delta V_{\text{amp}} - 0.53266 \Delta \omega) \Delta \theta \\
+ 4.23226 \Delta V_{\text{amp}}^2 - 1.82448 \Delta \omega^2 - 7.41318 \Delta V_{\text{amp}} \Delta \omega
\end{bmatrix}
\]

and

\[
D(x, \omega, V_{\text{amp}}) - x^* = \\
\begin{cases}
\text{Id} & \text{when } \Delta q_2 \leq 0 \\
0.237147 \Delta q_2 \\
-1.55767 \sqrt{\Delta q_2} - 0.189248 \Delta q_2 + \Delta \theta \\
+ (0.206574 \Delta \theta - 1.21057 \Delta V_{\text{amp}} - 3.1932 \Delta \omega) \sqrt{\Delta q_2}
\end{cases}
\] when \( \Delta q_2 > 0 \)

(53)

where we have omitted terms of order three or higher in \( \sqrt{\Delta q_2}, \Delta \theta, \Delta \omega, \) and \( \Delta V_{\text{amp}}. \)

Figures 7, 8, and 9 show the predictions of the local map \( \mathbf{P} = \tilde{P} \circ D \) on the bifurcations associated with variations in \( \Delta V_{\text{amp}} \) for select values of \( \Delta \omega \) corresponding to the schematic transition scenarios across \( \Gamma_I, \Gamma_{II}, \) and the boundary point between \( \Gamma_I \) and \( \Gamma_{II} \) as described in Section 2. While the predicted results are in excellent agreement with the simulated data for the original dynamical system, we recognize that the former were obtained at a fraction of the computing time required to generate the latter.

The local map shown above is valid on an open neighborhood of the co-dimension-two point at the boundary between \( \Gamma_I \) and \( \Gamma_{II} \). It follows that it should be able to capture the saddle-node bifurcation curve associated with \( \Gamma_I \) and the period-doubling bifurcation curve associated with \( \Gamma_{II} \). In fact, the explicit nature of the local map allows the determination of the bifurcation curves as Taylor expansions in the perturbation \( \Delta \omega \). Specifically, we find that
Fig. 7. Comparison of the bifurcation results predicted from the local map with those obtained through numerical integration, when $\omega$ is at the boundary between $\Gamma_I$ and $\Gamma_{II}$. Panel a) shows variation of $\Delta q_2$ with the applied voltage, where the impacting motions are in black and the nonimpacting motions are in gray. We note that the predicted results are indistinguishable from the numerical results. Panel b) shows the dependence of the real and imaginary parts of the eigenvalues of the periodic impacting orbit on the applied voltage as obtained from numerical simulations (solid) and as predicted by the local mapping (dashed).

Fig. 8. Comparison of the bifurcation results predicted from the local map with those obtained through numerical simulation, when $\omega = 0.4888$. Panel a) shows variation of $\Delta q_2$ with the applied voltage, where the impacting motions are in black, the nonimpacting motions are in gray, solid curves correspond to stable periodic motions, and dashed curves correspond to unstable periodic motions. Panels b) and c) show the dependence of the eigenvalues of the periodic impacting orbit on the applied voltage as obtained from numerical simulations (dots for stable and circles for unstable) and as predicted by the local mapping (solid for stable and dashed for unstable).

the saddle-node bifurcation curve is given by

$$\Delta V_{\text{amp}} = 0.224982 \Delta \omega + 17.7774 \Delta \omega^2 + \mathcal{O}(\Delta \omega^3), \Delta \omega > 0$$

(55)
Fig. 9. Comparison of the bifurcation results obtained through numerical simulation (Panel a)) with those predicted from the local map (Panel b)), when $\omega = 0.485$. The impacting motions are in black, the nonimpacting motions are in gray, solid curves correspond to stable periodic motions, dashed curves correspond to unstable periodic motions, and black regions correspond to chaotic attractors. Panels c) and d) show the dependence of the eigenvalues of the periodic impacting orbit on the applied voltage as obtained from numerical simulations (dots for stable and circles for unstable) and as predicted by the local mapping (solid for stable and dashed for unstable).

and the period-doubling curve is given by

$$\Delta V_{\text{amp}} = 0.224982 \Delta \omega + 1557.35 \Delta \omega^2 + O(\Delta \omega^3), \Delta \omega < 0.$$  \hspace{1cm} (56)

Figures 10 a) and b) compare the predicted bifurcation points using the local map and the predicted bifurcation curves using the truncated Taylor expansions, respectively, with the actual bifurcation points obtained from direct numerical simulation of the original dynamical system. Again, it is clear that the local map provides a very accurate description of the dynamics in the vicinity of the grazing periodic trajectory in state and parameter space.

The grazing periodic trajectory at the co-dimension-two point at the boundary between $\Gamma_{\text{II}}$ and $\Gamma_{\text{III}}$ corresponds to

$$\omega^* \approx 0.480271, V_{\text{amp}}^* \approx 0.176796 \text{ and } x^* \approx \left(0.5, 0.286519\right)^T.$$  \hspace{1cm} (57)
Fig. 10. Bifurcation curves obtained numerically (Panels a) and b)), compared to those predicted from the local map a)), and those predicted from the Taylor expansions b)). In both panels, triangles, circles, and "+" correspond to the numerically obtained grazing, period doubling, and saddle node bifurcation points, respectively; the solid lines, dashed lines, and dotted lines correspond to the predicted grazing, period doubling, and saddle node bifurcation points, respectively.

Again, following the methodology in the previous section and the comments made in the last paragraph of the previous section, we find

\[
\tilde{P}(x, \omega, V_{\text{amp}}) - x^* = \\
\begin{pmatrix}
0.94297 \Delta q_2 + 0.0512257 \Delta \theta + 0.809064 \Delta V_{\text{amp}} + 0.00715561 \Delta \omega \\
+ (0.138282 \Delta q_2 + 0.0544217 \Delta \theta + 1.24226 \Delta V_{\text{amp}} - 0.431478 \Delta \omega) \Delta q_2 \\
- (0.105538 \Delta \theta - 0.426753 \Delta V_{\text{amp}} + 0.888627 \Delta \omega) \Delta \theta \\
+ 4.24501 \Delta V_{\text{amp}}^2 - 2.288693 \Delta \omega^2 - 1.16325 \Delta V_{\text{amp}} \Delta \omega \\
0.192815 \Delta q_2 + 0.826809 \Delta \theta + 1.5879 \Delta V_{\text{amp}} + 6.00327 \Delta \omega \\
+ (0.724366 \Delta q_2 + 3.29973 \Delta V_{\text{amp}} - 0.342789 \Delta \omega - 0.136173 \Delta \theta) \Delta q_2 \\
- (0.0565448 \Delta \theta + 2.32779 \Delta V_{\text{amp}} + 1.15147 \Delta \omega) \Delta \theta \\
+ 7.230497180635611 \Delta V_{\text{amp}}^2 - 3.61189 \Delta \omega^2 - 6.14775 \Delta V_{\text{amp}} \Delta \omega
\end{pmatrix}
\]  

(58)
and

\[
D(x, \omega, V_{\text{amp}}) - x^* = \begin{cases} 
\text{Id} \\
\begin{pmatrix} 
0.238028 \Delta q_2 \\
-1.58753 \sqrt{\Delta q_2} - 0.204934 \Delta q_2 + \Delta \theta \\
+ (0.219773 \Delta \theta - 1.92188 \Delta V_{\text{amp}} - 3.30549 \Delta \omega) \sqrt{\Delta q_2}
\end{pmatrix}
\end{cases}
\]

when \( \Delta q_2 \leq 0 \)

\[
\begin{cases} 
\text{Id} \\
\begin{pmatrix} 
0.238028 \Delta q_2 \\
-1.58753 \sqrt{\Delta q_2} - 0.204934 \Delta q_2 + \Delta \theta \\
+ (0.219773 \Delta \theta - 1.92188 \Delta V_{\text{amp}} - 3.30549 \Delta \omega) \sqrt{\Delta q_2}
\end{pmatrix}
\end{cases}
\]

when \( \Delta q_2 > 0 \),

(59)

where we have again omitted terms of order three or higher in \( \sqrt{\Delta q_2} \), \( \Delta \theta \), \( \Delta \omega \), and \( \Delta V_{\text{amp}} \).

Figures 11 and 12 show the excellent agreement between the predictions of the local map \( P = \tilde{P} \circ D \) and the result of direct numerical simulations of the original dynamical system. Again, we may choose to use the local map to predict the locus of the saddle-node and global crisis bifurcations. Figure 13 shows predicted bifurcation points and corresponding bifurcation points obtained using the simulated data. The observation made previously regarding computation time applies again. Indeed, while the predicted bifurcation values were obtained in a matter of minutes, the simulated data required several days of computations. Using the predicted data as initial conditions for the numerical simulation, the computation time was reduced to a couple of hours.

![Graph](image1)

![Graph](image2)

Fig. 11. Comparison of the bifurcation results obtained through numerical simulation (Panel a)) with those predicted from the local map (Panel b)), when \( \omega = 0.4803 \). The impacting motions are in black, the nonimpacting motions are in gray, solid curves correspond to stable periodic motions, dashed curves correspond to unstable periodic motions, and black regions correspond to chaotic attractors.
Fig. 12. Comparison of the bifurcation results obtained through numerical simulation (Panel a)) with those predicted from the local map (Panel b)), when $\omega = 0.4793$. The impacting motions are in black, the nonimpacting motions are in gray, solid curves correspond to stable periodic motions, dashed curves correspond to unstable periodic motions, and black regions correspond to chaotic attractors.

Fig. 13. Comparison of predicted bifurcation points with those obtained using the simulated data (solid line - numerically obtained grazing bifurcation points, dots - predicted grazing bifurcation points, + signs - numerically obtained crisis points, circles - predicted crisis points, * signs - numerically obtained saddle node bifurcation points, triangles - predicted saddle node bifurcation points.

5 Conclusion

In the previous sections, we formulated discrete maps that capture the dynamics of a model microelectromechanical impact actuator in the vicinity in state and parameter space of the transition between nonimpacting and impacting asymptotic dynamics. Specifically, using a discontinuity-mapping-based approach, we were able to achieve excellent agreement between the results of computationally-expensive direct numerical simulations of the impacting dynamics and the computationally cheap predictions from the associated dis-
crete maps. Indeed, the discrete maps faithfully describe the local dynamics while reducing the computation time by several orders of magnitude. As such the methodology provides an efficient tool in the analysis and design of proper function in impact microactuators. While the analysis was applied to the Mita actuator, only nominal modifications would be required to derive similar local maps to other impact microactuators.

Although we have only shown explicit formulae for the local map on neighborhoods of the two co-dimension-two points, similar maps can easily be derived on neighborhoods of the co-dimension-one points. It is interesting to observe that the predictions of such local maps when truncated at first order in \( \sqrt{\Delta q_2}, \Delta \theta, \Delta \omega, \) and \( \Delta V_{\text{amp}} \) agree qualitatively with the numerical simulations for small perturbations near the co-dimension-one points, but that the second-order terms are necessary to capture the bifurcation behavior near the co-dimension-two points.

Following Zhao et al [24], the above discussion focused on the interval \( 0.46 < \omega < 0.51 \), which contains the primary resonance (recall that the frequency of the forcing term is twice that of the voltage oscillations). A study of the system behavior outside of this interval is currently underway. Preliminary results show the existence of additional co-dimension-two grazing bifurcations, e.g., involving the coincidence of a smooth period-doubling bifurcation with grazing contact and associated transition scenarios. These will be the subject of a separate discussion.

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7 Appendix

To map out the grazing curve \( \Gamma \) in the \( \omega - V_{\text{amp}} \) parameter space, we desire a numerical method for locating points of grazing contact \( x^* \in \mathcal{D} \cap \mathcal{P} \), such that \( \mathcal{P}(x^*, \omega, V_{\text{amp}}) = x^* \), for the Poincaré map \( \mathcal{P} \) associated with \( \mathcal{P} \).

Denote by \( \Phi \) the flow corresponding to motion in the submanifold \( \mathcal{S} \) with vector field \( f = f_{\text{stick}} \). Consider the function

\[
g(t, x, \omega, V_{\text{amp}}) = h^P(\Phi(x, t; \omega, V_{\text{amp}}))
\]
for some $x$. Then,

\[
g_t(t, x, \omega, V_{\text{amp}}) = h_i^D(y) \cdot f^i(y, \omega, V_{\text{amp}}) \bigg|_{y=\Phi(x, t; \omega, V_{\text{amp}})} (61)
\]

\[
g_{tt}(t, x, \omega, V_{\text{amp}}) = \begin{bmatrix} h_{ij}^D(y) \cdot f^i(y, \omega, V_{\text{amp}}) \\ + h_i^D(y) \cdot f^i_j(y, \omega, V_{\text{amp}}) \end{bmatrix} \cdot f^j(y, \omega, V_{\text{amp}}) \bigg|_{y=\Phi(x, t; \omega, V_{\text{amp}})} (62)
\]

Specifically, suppose that for some choice of parameters $\omega = \omega^*$ and $V_{\text{amp}} = V_{\text{amp}}^*$, the point $x^* \in P$ lies on a periodic trajectory with period $T$ of the nonimpacting system and that $t = 0$ is a simple local maximum of the function $h^P\left(\Phi(x^*, t; \omega^*, V_{\text{amp}}^*)\right)$. It follows that

\[
g_t(0, x^*, \omega^*, V_{\text{amp}}^*) = h_i^D(x^*) \cdot f^i(x^*, \omega^*, V_{\text{amp}}^*) = 0 \tag{63}
\]

and

\[
g_{tt}(0, x^*, \omega^*, V_{\text{amp}}^*) = \begin{bmatrix} h_{ij}^D(x^*) \cdot f^i(x^*, \omega^*, V_{\text{amp}}^*) \\ + h_i^D(x^*) \cdot f^i_j(x^*, \omega^*, V_{\text{amp}}^*) \end{bmatrix} \cdot f^j(x^*, \omega^*, V_{\text{amp}}^*) < 0. \tag{64}
\]

Now consider the function

\[
F(x, t, \omega, V_{\text{amp}}) = h^P\left(\Phi(x, t; \omega, V_{\text{amp}})\right) = h_i^D\left(\Phi(x, t; \omega, V_{\text{amp}})\right) \cdot f^i\left(\Phi(x, t; \omega, V_{\text{amp}})\right). \tag{65}
\]

Then,

\[
F(x^*, T, \omega^*, V_{\text{amp}}^*) = g_t(0, x^*, \omega^*, V_{\text{amp}}^*) = 0 \tag{66}
\]

and

\[
F_t(x^*, T) = g_{tt}(0, x^*, \omega^*, V_{\text{amp}}^*) < 0. \tag{67}
\]

The implicit function theorem implies the existence of a unique smooth function $\tau(x)$ for $x \approx x^*$, $\omega \approx \omega^*$, and $V_{\text{amp}} \approx V_{\text{amp}}^*$, such that

\[
\tau(x^*, \omega^*, V_{\text{amp}}^*) = T \tag{68}
\]

and

\[
h^P\left(\Phi(x, \tau(x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}})\right) = F \left( x, \tau(x, \omega, V_{\text{amp}}), \omega, V_{\text{amp}} \right) \equiv 0. \tag{69}
\]

Using $\tau(x)$, we can now uniquely define the Poincaré mapping $P$ corresponding to $P$ by

\[
P(x, \omega, V_{\text{amp}}) = \Phi(x, \tau(x, \omega, V_{\text{amp}}); \omega, V_{\text{amp}}) \tag{70}
\]

for $x \approx x^*$, $\omega \approx \omega^*$, and $V_{\text{amp}} \approx V_{\text{amp}}^*$. 
Now consider the vector valued function
\[ F(x, \omega, V_{\text{amp}}) = \begin{pmatrix} P(x, \omega, V_{\text{amp}}) - x \\ h^D(x) \end{pmatrix}. \] (71)

Zeros of this function clearly correspond to fixed points of the Poincaré mapping \( P \) that lie in \( \mathcal{D} \cap \mathcal{P} \), i.e., grazing periodic trajectories. Indeed, provided that the matrix
\[ F_{(x,V_{\text{amp}})} = \begin{pmatrix} P_x(x, \omega, V_{\text{amp}}) - Id & P_{,\omega}(x, \omega, V_{\text{amp}}) \\ h^D_{,x}(x) & 0 \end{pmatrix} \] (72)
is invertible, we may implement a Newton scheme to locate zeros of \( F \) and thus \( x^*(\omega) \) and \( V_{\text{amp}}^*(\omega) \). This is the method employed in generating the grazing curve \( \Gamma \) in Figure 4.

References


