A Simplified Axiomatic Approach to Ambiguity Aversion

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Abstract

This paper takes the Anscombe-Aumann framework with horse and roulette lotteries, and applies the Savage axioms to the horse lotteries and the von Neumann-Morgenstern independence axiom to the roulette lotteries. The resulting representation of preferences yields a subjective probability measure over states and two utility functions, one governing risk attitudes and one governing ambiguity attitudes. The model is able to accommodate the Ellsberg paradox and preferences for reductions in ambiguity.

Keywords: Ambiguity; Savage axioms; Anscombe-Aumann framework; independence axiom; Ellsberg paradox

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1 Introduction

One of the famous problems that highlights the difference between risk and ambiguity (or uncertainty) is the two-color Ellsberg problem (see Ellsberg, 1961). A decision maker is faced with two urns. The first urn contains 50 red and 50 yellow balls, and the second contains 100 balls but in an unknown mixture of red and yellow. The decision maker will be paid $10 if she can draw a red ball and must choose from which urn to draw. The first urn generates a known payoff distribution, so it is risky, but the second urn generates an unknown payoff distribution, so it is ambiguous. Subjects systematically avoid the ambiguous urn in favor of the risky urn, thereby creating the need for a model of choice behavior which can accommodate the distinction between risk and ambiguity.

Ambiguity aversion is inconsistent with the standard subjective expected utility models, and so those models must be generalized. Two axiomatizations of subjective expected utility stand out in the literature. Savage (1954) assumes that the state space is infinite and that the object of choice is an act, which is a mapping from states of nature to payoffs. Anscombe and Aumann (1963) assume a finite state space and that the object of choice is a horse lottery, which is a mapping from states of nature into roulette lotteries, or probability distributions. In this paper I generalize subjective expected utility by assuming Savage's infinite state space, Anscombe and Aumann's formulation of horse and roulette lotteries, and applying Savage's axioms to preferences over horse lotteries instead of acts. In addition, I apply the familiar von Neumann-Morgenstern independence axiom when preferences are restricted to roulette lotteries. This very simple approach yields a preference representation of the form [1]

$$W(h) = \int_S w\left(\int_X u(x)d(h_s(x))\right)d\mu(s), \tag{1}$$

where the state of nature $s \in S$ determines the objective probability distribution (roulette lottery) h_s over payoffs $x \in X$, μ is the subjective probability distribution over states of nature, u is a von Neumann-Morgenstern utility function governing attitudes toward risk, and w is another utility function, this time governing attitudes toward ambiguity. The overall preference function W is defined over horse lotteries $h \in \mathcal{H}$, which are assignments of objective probability distributions to states of nature. We call preferences that have the representation in (1) second-order expected utility preferences.

Other researchers have used two methods to generalize the subjective expected utility model to allow for ambiguity aversion. One approach is based on nonadditive probabilities, and examples can be found in Gilboa (1987), Schmeidler (1989), Gilboa and Schmeidler (1989), Sarin and Wakker (1992), and Maccheroni, Marinacci, and Rustichini (2006). The alternative approach, which has generated considerable recent interest, is based on second-order probabilities, or subjective probability distributions defined over objective probability distributions. An early example of this research can be found in Hazen (1987),[2] and more recent work can be found in Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), Strzalecki (2007), Ahn (2008), Chew and Sagi (2008), and Ergin and Gul (2009). All of these papers generate functional forms similar to that in (1), but with different interpretations and different constructions. The interpretation here is not as a true second-order probability, since the subjective probability distribution is defined over states of nature and not other (objective)

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[1] This functional specification is also proposed by Kreps and Porteus (1978) for the analysis of dynamic choices under risk. The specification here was first proposed in Neilson (1993).

[2] Hazen and Lee (1991) show how Hazen's model accommodates evidence such as the Ellsberg paradox.

probability distributions. The key to the construction here is that states correspond to objective probability distributions, as in the Anscombe-Aumann approach, and this paper's contribution to the literature is the simplicity of the axiomatic framework. In particular, no new axioms are proposed, only a different application of the old axioms.

Section 2 sets up the Anscombe-Aumann framework, identifies the axioms, and provides the representation theorem. Section 3 shows that the second-order expected utility representation in expression (1) can easily accommodate ambiguity averse behavior when the function w is concave and the subjective distribution μ is uniform. Section 4 offers a brief conclusion.

2 The representation theorem

The model adopts the roulette and horse lottery framework of Anscombe and Aumann (1963). The bounded interval X is the payoff space, and $\Delta(X)$ is the set of all probability distributions over X. Members of $\Delta(X)$ are also called *roulette lotteries*. Let S be the set of states of the world, with generic element s. Define Σ to be the set of all subsets of S, with generic element E, which is interpreted as an event. Savage (1954) defines an *act* as a mapping from S to X, while Anscombe and Aumann define a *horse lottery* as a mapping from S to $\Delta(X)$, that is, a mapping assigning a roulette lottery to every state. The resolution of an act is an outcome in the payoff space, while the resolution of a horse lottery is a roulette lottery, which is a probability distribution over payoffs. Let $\mathcal{H}$ denote the set of all horse lotteries.

The key to this paper is applying the Savage axioms to horse lotteries instead of acts. To do so, assume that the individual has a preference ordering $\succsim$ defined over $\mathcal{H}$. In what follows, $f, f', h, h' \in \mathcal{H}$ are horse lotteries, $\pi, \pi', \rho, \rho' \in \Delta(X)$ are roulette lotteries, and $E, E', E_i \in \Sigma$ are events. Abusing notation when the context is clear, the roulette lottery π is also a degenerate horse lottery assigning the same probability distribution to every state in S, that is, $h_s = \pi$ for all $s \in S$. These degenerate horse lotteries are called *constant lotteries*. The set E^c is the complement of E in S, that is, $S \setminus E$. A set E is *null* if $h \sim f$ whenever $h_s = f_s$ for all $s \in E^c$, and where $\sim$ is the indifference relation. It is said that $h = f$ *on* E if $h_s = f_s$ for all $s \in E$. It is said that $h \succeq f$ *given* E if and only if $h' \succeq f'$ whenever $h_s = h'_s$ for $s \in E$, $f_s = f'_s$ for $s \in E$, and $h'_s = f'_s$ for all $s \in E^c$.

We use the following axioms over $\succsim$. Axioms A1 - A7 are the Savage axioms, and axiom A8 applies von Neumann and Morgenstern's independence axiom to constant horse lotteries, which are simply probability distributions.

A1: (Ordering) – $\succeq$ is complete and transitive.

A2: (Sure-thing principle) – If $f = f'$ and $h = h'$ on E, and $f = h$ and $f' = h'$ on E^c, then $f \succeq h$ if and only if $f' \succeq h'$.

A3: (Eventwise monotonicity) – If E is not null and if $f = \pi$ and $h = \rho$ on E, then $f \succeq h$ given E if and only if $\pi \succeq \rho$

A4: (Weak comparative probability) – Suppose that $\pi \succeq \rho$, $f = \pi$ on E, $f = \rho$ on E^c, $h = \pi$ on E', and $h = \rho$ on E'^c, and suppose that $\pi' \succeq \rho'$, $f' = \pi'$ on E, $f' = \rho'$ on E^c, $h' = \pi'$ on E', and $h' = \rho'$ on E'^c. Then $f \succeq h$ if and only if $f' \succeq h'$.

A5: (Nondegeneracy) – $\pi \succ \rho$ for some $\pi, \rho \in \Delta(X)$.

A6: (Small event continuity) – If $f \succ h$, for every $\pi \in \Delta(X)$ there is a finite partition of S such that for every E_i in the partition, if $f' = \pi$ on E_i and $f' = f$ on E_i^c then $f' \succ h$, and if $h' = \pi$ on E_i and $h' = h$ on E_i^c then $f \succ h'$.

A7: (Uniform monotonicity) – For all $E \in \Sigma$ and for all $\pi \in h(E)$, if $f \succeq \pi$ given E, then $f \succeq h$ given E. If $\pi \succeq f$ given E, then $h \succeq f$ given E.

A8: (Independence over risk) – $\pi \succeq \pi'$ if and only if $a\pi + (1-a)\rho \succeq a\pi' + (1-a)\rho$ for all $\rho \in \Delta(X)$ and all scalars $a \in (0, 1)$.

Axioms A1 - A7 are the standard Savage axioms modified so that they govern preferences over horse lotteries instead of preferences over acts. The main difference between these axioms and Savage's, then, is that here probability distributions in $\Delta(X)$ replace outcomes in X.

It is worth differentiating the sure-thing principle A2 from the independence over risk axiom A8. Consider the following application of axiom A2 to roulette lotteries, in which the horse lotteries $f, f', h, h' \in \mathcal{H}$ yield the following probability distributions in states $E, E^c \in \Sigma$, where $\pi, \rho, \sigma, \tau \in \Delta(X)$:

	E	E^c
f	π	σ
f'	π	τ
h	ρ	σ
h'	ρ	τ

The sure-thing principle states that $f \succeq h$ if and only if $f' \succeq h'$, that is, preferences only depend on states in which the two horse lotteries being considered have different outcomes. This has the same spirit as the independence axiom A8, but with one key difference. Here the mixture f of π and σ is *not* a probability mixture, and thus it is not in $\Delta(X)$. Consequently, axiom A2 does not imply axiom A8. However, axiom A3 guarantees that preferences on constant horse lotteries are identical to preferences over the corresponding roulette lotteries.

Theorem 1 *Preferences satisfy A1 - A8 if and only if there exists a unique probability* *measure* $\mu : \Sigma \to [0,1]$*, a function* $u : X \to \mathbb{R}$*, and a function* $w : \mathbb{R} \to \mathbb{R}$ *such that for all* $f, h \in \mathcal{H}$*,* $h \succeq f$ *if and only if*

$$\int_S w\left(\int_X u(x)d(h_s(x))\right)d\mu(s) \geq \int_S w\left(\int_X u(x)d(f_s(x))\right)d\mu(s).$$

Moreover, the function u *is unique up to increasing affine transformations, and for a given specification of* u *the function* w *is unique up to increasing affine transformations.*

Proof. Proof of the "if" direction is standard. For the "only if" direction, by the Savage axioms A1 - A7, there exists a unique probability measure $\mu : \Sigma \to [0,1]$, and a function $v : \Delta(X) \to \mathbb{R}$ such that the preference ordering $\succsim$ is represented by the functional

$$W(h) = \int_S v\left(h_s\right)d\mu(s). \tag{2}$$

Furthermore, μ is unique and v is unique up to increasing affine transformations. By axiom A8, $\succsim$ restricted to constant horse lotteries can be represented by

$$V(\pi) = \int_X u(x)d\pi(x), \tag{3}$$

where u is unique up to increasing affine transformations. Axiom A3 implies that $V(\pi)$ and $v(\pi)$ must represent the same preferences over roulette lotteries, and so there exists a monotone function $w : \mathbb{R} \to \mathbb{R}$ such that

$$v(\pi) = w(V(\pi)). \tag{4}$$

Since v is unique up to increasing affine transformations, so is w for a given specification of u. Substituting expression (3) into (4) and then (4) into (2) yields

$$W(h) = \int_S w\left(\int_X u(x) dh_s(x))\right) d\mu(s). \tag{5}$$

■

It is worth pointing out why the axioms do not get us all the way to subjective expected utility (and ambiguity neutrality). The key is that axioms A1 - A7 only link the event space Σ to the roulette lottery space $\Delta(X)$, and not all the way to the payoff space X. Axiom A8 places structure on the link between the roulette lottery space and the payoff space, but not enough to provide that missing link. Consequently, risk attitudes and ambiguity attitudes remain separated.

3 Ambiguity aversion

The second-order expected utility specification in (1) easily accommodates ambiguity attitudes as revealed in patterns such as the Ellsberg paradox. First consider the two-color paradox described in the introduction, where an individual is paid \$10 for drawing a red ball from one of two urns. States correspond to the number of red balls in the ambiguous urn, and for ease of exposition assume that this is a continuous variable between 0 and 100. Betting on the unambiguous urn corresponds to a constant horse lottery f which yields a 50:50 chance of \$10 in every state. Betting on the ambiguous urn corresponds to a horse lottery h which yields a probability $s/100$ of winning \$10. Since there are only two payoffs, we can normalize the von Neumann-Morgenstern utility function u so that $u(10) = 1$ and $u(0) = 0$. If μ is uniform over states we get

$$W(f) = w\left(\frac{1}{2}\right)$$

and

$$W(h) = \frac{1}{100} \int_0^{100} w\left(\frac{s}{100}\right) ds.$$

In the Ellsberg example individuals tend to choose f over h, and Jensen's inequality implies that $W(f) \geq W(h)$ if w is concave.

Uniform μ and concave w can also explain the three-color Ellsberg paradox. In this paradox a single urn contains 90 balls, 30 of which are red and the remaining 60 an unknown mixture of yellow and black. When individuals are given a chance to win \$100 if they draw a red ball or \$10 if they draw a yellow ball, they tend to bet on a red ball. When individuals are given a chance to win \$10 if they draw either a red or black ball, or \$10 if they draw either a yellow or black ball, they tend to bet on the yellow and black combination. In a subjective expected utility context the first choice suggests that fewer than 30 of the balls are yellow, and the second choice suggests that fewer than 30 of the balls are black; hence the paradox.

Let s define the state according to the number of yellow balls, so that $60 - s$ is the number of black balls, and normalize u in the same way as above. The constant horse lottery f is a bet on drawing a red ball, the horse lottery f' is a bet on drawing a red or black ball, the horse lottery h is a bet on drawing a yellow ball, and the constant horse lottery h' is a bet on drawing a yellow or black ball. The Ellsberg preferences have $f \succ h$ but $h' \succ f'$. One can compute

$$\begin{aligned}
W(f) &= w\left(\frac{1}{3}\right), \\
W(h) &= \frac{1}{60} \int_0^{60} w\left(\frac{s}{90}\right) ds, \\
W(f') &= \frac{1}{60} \int_0^{60} w\left(\frac{90 - s}{90}\right) ds, \\
W(h') &= w\left(\frac{2}{3}\right).
\end{aligned}$$

If w is concave Jensen's inequality implies that $W(f) \geq W(h)$ and $W(h') \geq W(f')$.

Finally, consider an Ellsberg-like situation in which there are three urns, all of which contain a mixture of 10 red and yellow balls. Urn 1 is unambiguous and contains 50 red and 50 yellow balls. Urn 2 contains an unknown mixture of 100 balls, but with at least 30 red and at least 30 yellow balls. Urn 3 is completely ambiguous, containing a completely unknown mixture of 100 red and yellow balls. An individual can win \$10 for drawing a red ball from one of the urns. Intuition from the two-color Ellsberg paradox suggests that the individual would prefer urn 1 to urn 2 to urn 3. Letting f be the bet on urn 1, g the bet on urn 2, and h the bet on urn 3, and assuming uniform subjective probabilities, one can compute

$$\begin{aligned} W(f) &= w\left(\frac{1}{2}\right), \\ W(g) &= \frac{1}{40}\int_{30}^{70} w\left(\frac{s}{100}\right) ds, \\ W(h) &= \frac{1}{100}\int_{0}^{100} w\left(\frac{s}{100}\right) ds, \end{aligned}$$

and concave w implies the above preferences.

4 Conclusion

This paper applies old axioms from Savage (1954) and von Neumann and Morgenstern (1944) to an old choice framework developed by Anscombe and Aumann (1963) in which states of the world correspond to objective risks. The axioms lead to second-order expected utility preferences which consist of a subjective probability measure over states of the world, a utility function governing risk attitudes, and another utility function governing ambiguity attitudes. Concavity of the first utility function implies risk aversion, and concavity of the second is consistent with ambiguity aversion in the Ellsberg paradoxes when the subjective distribution over states is uniform. The second-order expected utility model is also consistent with the Kreps-Porteus (1978) model governing temporal uncertainty.

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