Econometrica, Vol. 55, No. 5 (September, 1987), 1139-1149

THE ROSS CHARACTERIZATION OF RISK AVERSION: STRENGTHENING AND EXTENSION

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This paper offers an interpretive comparison of the Arrow/Pratt and Ross characterizations of comparative risk aversion for expected utility maximizers. The tools used in this comparison are then applied to obtain a strengthening of the Ross characterization. This strengthened result is then extended to the case of general smooth non-expected utility preferences over probability distributions.

KEYWORDS: Risk aversion, expected utility, non-expected utility models.

1. INTRODUCTION

THE PURPOSE OF THIS PAPER is to continue and extend the Ross (1981) analysis of comparative risk aversion along both intuitive and analytical lines. Section 2 below offers an interpretive comparison of the Ross and standard Arrow/Pratt characterizations in a manner designed to highlight their similarities and differences. Section 3 presents a strengthening of the original Ross characterization along a few different lines. Section 4 extends this strengthened characterization to the case of general smooth non-expected utility preferences over probability distributions.

2. THE ARROW-PRATT AND ROSS MEASURES OF RISK AVERSION

One of the most useful results in the theory of individual behavior toward risk is the Arrow-Pratt characterization of comparative risk aversion for expected utility maximizers (Arrow (1963; 1974, Ch. 3), Pratt (1964)). This result states that the following conditions on a pair of twice-differentiable von Neumann-Morgenstern utility functions $U(\cdot)$ and $U^*(\cdot)$ are equivalent:[2]

(A.1) $\quad U^*(x) \equiv \rho(U(x))$ for some increasing concave function $\rho(\cdot)$,

(A.2) $\quad -U_{11}^*(x)/U_1^*(x) \geqslant -U_{11}(x)/U_1(x)$ for all x, and

[1] We are grateful to Eddie Dekel, David Kreps, Michael Rothschild, Joel Sobel, Hal White, and an anonymous referee for helpful comments on this material. Machina would like to acknowledge the National Science Foundation (Grant SES 83-08165) and the Alfred P. Sloan Foundation for financial support. After this paper was accepted for publication we became aware of the independent and related work of Professor Michel Demers of Carlton University, available in revised form as Demers (1987).

[2] Throughout this paper we assume that all von Neumann-Morgenstern utility functions (and later, all "local utility functions") are twice continuously differentiable with positive first derivatives. The subscripts "1" and "11" denote first and second derivatives, "$\tilde{\ }$" denotes that a variable is stochastic, $E[\cdot]$ denotes expectation, and terms such as $F_{\tilde{x},\tilde{\varepsilon}}(\cdot,\cdot)$, $F_{\tilde{x}+\tilde{\varepsilon}}(\cdot)$, and $F_{\tilde{\varepsilon}|x}(\cdot|\cdot)$ denote the joint (cumulative) distribution of $(\tilde{x}, \tilde{\varepsilon})$, the distribution of $\tilde{x}+\tilde{\varepsilon}$, and the conditional distribution of $\tilde{\varepsilon}$ given x, etc.

(A.3) if π^* and π solve $U^*(x-\pi^*)=\int U^*(\omega)\,dF_{x+\tilde{\varepsilon}}(\omega)$ and $U(x-\pi)=\int U(\omega)\,dF_{x+\tilde{\varepsilon}}(\omega)$ where $E[\tilde{\varepsilon}]=0$, then $\pi^*\geqslant\pi$,

and if $U(\cdot)$ and $U^*(\cdot)$ are both concave, these are equivalent to:

(A.4) if $E[\tilde{z}]\geqslant 0$ and $\bar{\alpha}^*$ and $\bar{\alpha}$ respectively maximize $\int U^*(\omega)\,dF_{x+\alpha\tilde{z}}(\omega)$ and $\int U(\omega)\,dF_{x+\alpha\tilde{z}}(\omega)$, then $\bar{\alpha}^*\leqslant\bar{\alpha}$.

Condition (A.1) states that the function $U^*(\cdot)$ is a "concave transformation" of $U(\cdot)$. (A.2) states that the "Arrow-Pratt measure of (absolute) risk aversion" $-U_{11}(x)/U_1(x)$ is everywhere at least as great for $U^*(\cdot)$ as for $U(\cdot)$. (A.3) states that if $U(\cdot)$ is just willing to pay a "risk premium" of π to avoid an actuarially neutral risk $\tilde{\varepsilon}$ about the wealth level x, then $U^*(\cdot)$ would be willing to pay at least this amount to avoid the same risk. (A.4) states that given the choice of dividing a (normalized) unit amount of wealth between a riskless asset with (gross) return x and a risky asset with return $x+\tilde{z}$ where $E[\tilde{z}]\geqslant 0$, $U(\cdot)$ would demand at least as much of the risky asset as would $U^*(\cdot)$.

Perhaps the best way to view the role played by the Arrow-Pratt ratio $-U_{11}(x)/U_1(x)$ is to consider the individual's marginal rate of substitution between risk and premium payments about an initial situation of certainty. Following the argument of Pratt (1964) and others, consider an individual with initial wealth x who faces an actuarially neutral risk $\sqrt{t}\cdot\tilde{\varepsilon}$ with var$[\tilde{\varepsilon}]=\sigma^2$ (so that var$[\sqrt{t}\cdot\tilde{\varepsilon}]=t\cdot\sigma^2$). The premium π that the individual would be just willing to pay to avoid this risk is given by the solution to

$$(1)\qquad U(x-\pi)=\int U(\omega)\,dF_{x+\sqrt{t}\cdot\tilde{\varepsilon}}(\omega).$$

Taking the Taylor expansion in π and t at $t=0$ (which implies $\pi=0$) yields

$$(2)\qquad -U_1(x)\cdot d\pi=\tfrac{1}{2}\cdot\sigma^2\cdot U_{11}(x)\cdot dt$$

so that we have

$$(3)\qquad \left.\frac{d\pi}{dt}\right|_{t=0}=-\tfrac{1}{2}\cdot\sigma^2\cdot\frac{U_{11}(x)}{U_1(x)}.$$

Thus for any risk $\tilde{\varepsilon}$, the greater an individual's Arrow-Pratt ratio, the greater his or her marginal rate of substitution between the scale-of-risk parameter t and the premium level π about an initial situation of certainty.

Although equation (3) may be used with the same level of rigor as any marginal rate of substitution in standard consumer theory (i.e. for the comparison of attitudes toward "small" risks), the key feature of the Arrow-Pratt result is that, as in the standard case, the pointwise comparison of these marginal rates of substitution (or equivalently, of the Arrow-Pratt ratios) is equivalent to the comparison of attitudes toward "large" (i.e. nondifferential) risks, as seen from conditions (A.3) and (A.4).

While the Arrow-Pratt characterization has proven tremendously useful in the theory of individual behavior toward risk,[3] Ross (1981) has pointed out that the risk premium and asset demand conditions (A.3) and (A.4) are both formulated with reference to situations of complete certainty: i.e. premiums for *complete* insurance against risk, and the allocation of wealth between risky and *completely* safe assets. However, as Ross has noted, the real world seldom affords such total security: most forms of insurance typically cover only some types of risks and not others, and in a world of price level uncertainty and bankruptcy, no asset, real or nominal, can be completely risk-free.

Accordingly, Ross (1981) (see also Ross (1979)) has developed an alternative characterization of risk aversion[4] which states that the following conditions on a pair of twice-differentiable risk averse (i.e. concave) utility functions $U(\cdot)$ and $U^*(\cdot)$ are equivalent:[5]

(R.1) $U^*(x) \equiv \lambda \cdot U(x) + G(x)$ for some positive constant λ and nonincreasing concave function $G(\cdot)$,

(R.2) $-U_{11}^*(x)/U_1^*(y) \geqslant -U_{11}(x)/U_1(y)$ for all x, y,[6] and

(R.3) if π^* and π solve $\int U^*(\omega)\, dF_{\tilde{x}-\pi^*}(\omega) = \int U^*(\omega)\, dF_{\tilde{x}+\tilde{\varepsilon}}(\omega)$ and $\int U(\omega)\, dF_{\tilde{x}-\pi}(\omega) = \int U(\omega)\, dF_{\tilde{x}+\tilde{\varepsilon}}(\omega)$ where $E[\tilde{\varepsilon}|x] \equiv 0$, then $\pi^* \geqslant \pi$,

and that they in turn imply:

(R.4) if $E[\tilde{z}|x] \geqslant 0$ and $\bar{\alpha}^*$ and $\bar{\alpha}$ respectively maximize $\int U^*(\omega)\, dF_{\tilde{x}+\alpha\tilde{z}}(\omega)$ and $\int U(\omega)\, dF_{\tilde{x}+\alpha\tilde{z}}(\omega)$, then $\bar{\alpha}^* \leqslant \bar{\alpha}$.

Conditions (R.3) and (R.4) differ from (A.3) and (A.4) in that the individual generally does not have any opportunity for complete certainty. The risk premium condition (R.3) assumes that initial wealth $\tilde{x}$ is itself random, and that the individual is at most able to insure against a conditionally actuarially neutral risk $\tilde{\varepsilon}$. The asset demand condition (R.4) involves an asset with random return $\tilde{x}$ and one with return $\tilde{x} + \tilde{z}$ with a higher mean but possibly greater risk.

The following argument, based on Ross (1981, pp. 625-626), helps illustrate how this formulation differs from that of Arrow and Pratt. We have seen how an individual's marginal rate of substitution between risk and premium payments about a *certain* initial wealth level depends upon the value of the term $-U_{11}(x)/U_1(x)$. Consider now an individual with *random* initial wealth $\tilde{x}$ who faces a conditionally actuarially neutral risk $\sqrt{t} \cdot \tilde{\varepsilon}$ with var $[\tilde{\varepsilon}|x] \equiv \sigma^2(x)$ (so that

[3] For additional applications of the Arrow-Pratt measure to the comparative statics of choice under uncertainty, see Diamond and Stiglitz (1974).

[4] For a weaker characterization of comparative risk aversion with random initial wealth involving the Arrow-Pratt measure, see Kihlstrom, Romer, and Williams (1974).

[5] As Ross (1981, p. 630) has noted, these conditions can only be satisfied by a pair of non-affinely equivalent utility functions over a bounded domain. Accordingly, we shall assume throughout that the supports of all relevant distributions lie within the appropriate bounded interval.

[6] Although Ross expressed this condition as $U_{11}^*(x)/U_{11}(x) \geqslant U_1^*(y)/U_1(y)$ for all x and y, we adopt this version in order to highlight the correspondence with the Arrow-Pratt ratio as well as to be able to apply the characterization to nonconcave utility functions.

$\operatorname{var}[\sqrt{t}\cdot\tilde{\varepsilon}|x]\equiv t\cdot\sigma^2(x)$). The premium π that the individual would be just willing to pay to avoid this risk is now given by the solution to

$$\int U(\omega)\,dF_{\tilde{x}-\pi}(\omega)=\int U(\omega)\,dF_{\tilde{x}+\sqrt{t}\cdot\tilde{\varepsilon}}(\omega). \tag{4}$$

Taking the Taylor expansion in π and t at $t=0$ (where again, $\pi=0$) yields

$$-\left[\int U_1(\omega)\,dF_{\tilde{x}}(\omega)\right]\cdot d\pi=\frac{1}{2}\cdot\left[\int \sigma^2(\omega)\cdot U_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)\right]\cdot dt \tag{5}$$

so that we have

$$\left.\frac{d\pi}{dt}\right|_{t=0}=-\frac{1}{2}\cdot\frac{\int\sigma^2(\omega)\cdot U_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int U_1(\omega)\cdot dF_{\tilde{x}}(\omega)}. \tag{6}$$

Thus an individual with utility function $U^*(\cdot)$ would be willing to pay as much to avoid this same infinitesimal risk about $\tilde{x}$ if and only if

$$-\frac{\int\sigma^2(\omega)\cdot U_{11}^*(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int U_1^*(\omega)\cdot dF_{\tilde{x}}(\omega)}\geqslant-\frac{\int\sigma^2(\omega)\cdot U_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int U_1(\omega)\cdot dF_{\tilde{x}}(\omega)}. \tag{7}$$

Changing the variable of integration in the denominators to ν, cross multiplying, and collecting terms yields that this condition is equivalent to

$$\iint\sigma^2(\omega)\cdot[U_{11}^*(\omega)\cdot U_1(\nu)-U_{11}(\omega)\cdot U_1^*(\nu)]\cdot dF_{\tilde{x}}(\omega)\cdot dF_{\tilde{x}}(\nu)\leqslant 0. \tag{8}$$

It is clear that this inequality will hold for all random initial wealth distributions and (infinitesimal) conditionally actuarially neutral risks if and only if $[U_{11}^*(\omega)\cdot U_1(\nu)-U_{11}(\omega)\cdot U_1^*(\nu)]\leqslant 0$ for all ω and ν, or equivalently, if and only if $-U_{11}^*(\omega)/U_1^*(\nu)\geqslant -U_{11}(\omega)/U_1(\nu)$ for all ω and ν, which is precisely condition (R.2). And as in the Arrow-Pratt case, the global nature of conditions (R.3) and (R.4) shows that the pointwise comparison of this ratio can be used for the comparison of attitudes toward large risks.

3. A STRENGTHENING OF THE ROSS CHARACTERIZATION

Our first result (Theorem 1) strengthens the Ross characterization in three ways. The first involves dropping the requirement of risk aversion for the equivalence of the first three Ross conditions. (We are grateful to Eddie Dekel for providing us with the proper generalization of (R.1) and a proof of its equivalence to (R.2).[7]) The second is that, conditional upon risk aversion, the asset demand condition (R.4) is not merely an implication of the first three conditions, but in fact equivalent to them, as in the Arrow-Pratt formulation. The third extension is the most substantive, and consists of replacing the assumption of a *certain* payment π in condition (R.3) with the more general case of a *stochastic* nonnegative premium payment.

[7] A pair of nonconcave utility functions satisfying conditions (i) through (iii) of the theorem over the interval [0, 50] is given by $U(x)\equiv x^3-30x^2+7500x$ and $U^*(x)\equiv x^3-60x^2+7500x$.

There are several reasons for the consideration of random premiums in the demand for insurance and related problems. Since real-world insurance premiums are usually fixed in nominal terms over a period where prices change randomly, *real* risk premiums are invariably stochastic. Second, if premiums are tax deductible and the marginal tax rate is nonconstant, a stochastic initial wealth implies a stochastic after-tax premium payment (although regular "market insurance" premiums are often not deductible, many of the expenses incurred in the "self-insurance" of income sources (e.g. Ehrlich and Becker (1972)) may be). Finally, the extension to the case of random premiums is very much in the spirit of Ross's program of making the analysis of behavior toward risk as independent as possible of the assumption of certainty.[8]

To see how the above "marginal rate of substitution" analysis may be extended to the case of random premiums, consider again an individual with initial wealth $\tilde{x}$ who faces a risk $\sqrt{t}\cdot\tilde{\varepsilon}$ with $E[\tilde{\varepsilon}|x]\equiv 0$ and $\operatorname{var}[\tilde{\varepsilon}|x]\equiv\sigma^2(x)$. Let $\tilde{\eta}$ be a nonnegative random variable with $E[\tilde{\eta}|x]\equiv h(x)$, and now let π be the "scale-of-premium" parameter which solves

$$\int U(\omega)\,dF_{\tilde{x}-\pi\cdot\tilde{\eta}}(\omega)=\int U(\omega)\,dF_{\tilde{x}+\sqrt{t}\cdot\tilde{\varepsilon}}(\omega). \tag{9}$$

Taking the Taylor expansion in π and t at $\pi=t=0$ yields

$$-\left[\int h(\omega)\cdot U_1(\omega)\,dF_{\tilde{x}}(\omega)\right]\cdot d\pi=\frac{1}{2}\cdot\left[\int \sigma^2(\omega)\cdot U_{11}(\omega)\,dF_{\tilde{x}}(\omega)\right]\cdot dt, \tag{10}$$

so that the marginal rate of substitution between the scale-of-risk and the scale-of-premium parameters about the initial wealth $\tilde{x}$ is

$$\left.\frac{d\pi}{dt}\right|_{t=0}=-\frac{1}{2}\cdot\frac{\int\sigma^2(\omega)\cdot U_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int h(\omega)\cdot U_1(\omega)\cdot dF_{\tilde{x}}(\omega)} \tag{11}$$

and the comparative condition between $U^*(\cdot)$ and $U(\cdot)$ becomes

$$-\frac{\int\sigma^2(\omega)\cdot U^*_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int h(\omega)\cdot U^*_1(\omega)\cdot dF_{\tilde{x}}(\omega)}\geqslant-\frac{\int\sigma^2(\omega)\cdot U_{11}(\omega)\cdot dF_{\tilde{x}}(\omega)}{\int h(\omega)\cdot U_1(\omega)\cdot dF_{\tilde{x}}(\omega)}. \tag{12}$$

Changing the variable of integration in the denominators to ν and cross multiplying yields that this condition is equivalent to

$$\iint h(\nu)\cdot\sigma^2(\omega)\cdot[U^*_{11}(\omega)U_1(\nu)-U_{11}(\omega)U^*_1(\nu)]\cdot dF_{\tilde{x}}(\omega)\cdot dF_{\tilde{x}}(\nu)\leqslant 0. \tag{13}$$

But since $h(\cdot)$ and $\sigma^2(\cdot)$ are arbitrary nonnegative functions, this inequality is again equivalent to the Ross condition (R.2).[9] The following theorem demonstrates that, as with the Arrow–Pratt and original Ross formulations, this marginal rate of substitution argument may be extended to the case of global risks:

[8] Although expressed in terms of moving up a risk-return tradeoff, Ross's Application 1 (1981, pp. 630–631) may also be interpreted as a random premium condition by replacing his variables $\tilde{x}$, $\tilde{\nu}$, and $\tilde{\varepsilon}$ with our variables $\tilde{x}-\pi\cdot\tilde{\eta}$, $\pi\cdot\tilde{\eta}$, and $\tilde{\varepsilon}$ respectively.

[9] Although this comparison of marginal rates of substitution about $\tilde{x}$ in fact only requires the nonnegativity of $h(x)\equiv E[\tilde{\eta}|x]$, our extension to large risks will require that $\tilde{\eta}$ itself be a nonnegative random variable.

THEOREM 1: *The following conditions are equivalent for a pair of twice-differentiable von Neumann-Morgenstern utility functions* $U(\cdot)$ *and* $U^*(\cdot)$ *on* $[0, M]$ *with* $U_1(\cdot), U_1^*(\cdot) > 0$:

(i) $U^*(x) \equiv \lambda \cdot U(x) + G(x)$ *for some* $\lambda > 0$ *and nonincreasing concave function* $G(\cdot)$ *satisfying* $G_{11}(x) \leq G_1(y) \cdot U_{11}(x)/U_1(y)$ *for all* $x, y \in [0, M]$;

(ii) $-U_{11}^*(x)/U_1^*(y) \geq -U_{11}(x)/U_1(y)$ *for all* $x, y \in [0, M]$;

(iii) *if* π^* *and* π *solve* $\int U^*(x)\, dF_{\tilde{x}-\pi^* \cdot \tilde{\eta}}(x) = \int U^*(x)\, dF_{\tilde{x}+\tilde{\varepsilon}}(x)$ *and* $\int U(x)\, dF_{\tilde{x}-\pi \cdot \tilde{\eta}}(x) = \int U(x)\, dF_{\tilde{x}+\tilde{\varepsilon}}(x)$ *where* $\tilde{\eta} \geq 0$ *and* $E[\tilde{\varepsilon}|x] \equiv 0$, *then* $\pi^* \geq \pi$;[10]

and if both $U(\cdot)$ *and* $U^*(\cdot)$ *are risk averse, these are in turn equivalent to*

(iv) *if* $E[\tilde{z}|x] \geq 0$ *and* $\bar{\alpha}^*$ *and* $\bar{\alpha}$ *respectively maximize* $\int U^*(\omega)\, dF_{\tilde{x}+\alpha\tilde{z}}(\omega)$ *and* $\int U(\omega)\, dF_{\tilde{x}+\alpha\tilde{z}}(\omega)$, *then* $\bar{\alpha}^* \leq \bar{\alpha}$.

Since all risk averse expected utility maximizers are "diversifiers" (see below), this theorem will follow from Theorem 2 below.

4. EXTENSION TO NON-EXPECTED UTILITY PREFERENCES

The argument of the previous sections as well as Ross (1981) concerned how conditions sufficient to compare attitudes toward *differential* risks about a random or nonrandom initial wealth were in fact sufficient to compare attitudes toward *global* risks. In this section we consider a different type of extension, namely from preferences that satisfy the expected utility property of "linearity in the probabilities" to general smooth preferences over probability distributions, i.e. those which are only "locally linear" in the probabilities, as studied for example by Allen (1987), Dekel (1986), Epstein (1985), and Machina (1982a, 1982b).

Specifically, we adopt the L^1 norm $\|F^* - F\| \equiv \int |F^*(x) - F(x)|\, dx$ over the set $D[0, M]$ of all cumulative distribution functions over $[0, M]$, and assume that the preference functional $V(F)$ is everywhere Fréchet differentiable with respect to $F(\cdot)$.[11] In Machina (1982a) this was shown to imply the existence of a "local utility function" $U(\cdot; F_0)$ at each distribution $F_0(\cdot)$ such that

$$V(F^*) - V(F_0) = \int U(\omega; F_0)[dF^*(\omega) - dF_0(\omega)] + o(\|F^* - F_0\|) \tag{14}$$

for all $F^*(\cdot)$, where $o(\cdot)$ denotes a function which is zero at zero and of higher order than its argument. Thus, just as in ordinary calculus, the difference $V(F^*) - V(F_0)$ can be expressed as the sum of a first order (i.e. linear) term and a higher order term, where the linear term can be represented as the difference in the expectations of the function $U(\cdot; F_0)$ with respect to the distributions $F^*(\cdot)$ and

[10] Note that since conditions (iii) and (R.3) are both equivalent to (R.2), it follows that $U^*(\cdot)$ will be willing to pay at least as much as $U(\cdot)$ in all *random* premium situations if and only if it is willing to pay as much as $U(\cdot)$ in all *certain* premium situations. This particular equivalence, however, will not extend to the case of non-expected utility preferences.

[11] The main result of this section (Theorem 2) can alternatively be derived using a weaker concept of differentiability along the lines of Chew (1983), Chew, Karni, and Safra (1986), etc.

$F_0(\cdot)$. Since the higher order term will disappear in the evaluation of any differential shift from $F_0(\cdot)$ to $F^*(\cdot)$, it follows that all of the *local* properties of preferences at $F_0(\cdot)$ are determined by the properties of the local utility function $U(\cdot; F_0)$ in the same manner as in expected utility theory. For example, $V(\cdot)$ will be averse to all differential mean-preserving increases in risk about $F_0(\cdot)$ if and only if $U(x; F_0)$ is a concave function of x. In Machina (1982a) it was also shown how most of the basic concepts, tools, and techniques of expected utility analysis may be *globally* extended in a similar manner. Thus, for example, $V(\cdot)$ will be averse to all *large* mean-preserving increases in risk if and only if $U(x; F)$ is concave in x for *each* $F(\cdot)$.

It therefore follows that the "marginal rate of substitution" arguments of the previous sections will generalize to the case of (smooth) non-expected utility preferences with the Arrow–Pratt and Ross ratios replaced by $-U_{11}(x; F_0)/U_1(x; F_0)$ and $-U_{11}(x; F_0)/U_1(y; F_0)$ respectively. And except for a technical modification, the global results in Theorem 1 can be similarly extended.

This modification concerns the precise role played by the assumption of risk aversion in the asset demand condition (iv) of Theorem 1. This role is *not* to ensure that the individuals will desire at least some of the less risky asset $\tilde{x}$ (in other words, that $\tilde{\alpha} \leqslant 1$), but rather to ensure that preferences are quasiconcave in asset holdings so that the proper comparative static response (iv) will obtain. As seen in the analyses of Tobin (1958, Fig. 6), Machina (1982a, 1982b), and especially Dekel (1986), the behavioral properties of risk aversion and quasiconcavity in asset holdings are in fact independent for general non-expected utility maximizers. Accordingly, we replace the assumption of risk aversion by the following condition:[12]

DEFINITION: An individual is said to be a *diversifier* if, for all $\tilde{x}$, $\tilde{z}$ such that $E[\tilde{z}|x] \geqslant 0$ for all x, his or her preferences over the set of random wealths $\{\tilde{x}+\alpha\tilde{z}\}_\alpha$ are strictly quasiconcave in α.

In the following, the term "smooth" denotes that the derivatives $U_1(x; F)$, $U_{11}(x; F)$, $U_1^*(x; F)$, and $U_{11}^*(x; F)$ vary continuously in $F(\cdot)$. Given these definitions, our extension of the Ross characterization to general smooth non-expected utility preferences is given by:

THEOREM 2: *The following conditions are equivalent for a pair of smooth Fréchet differentiable preference functionals $V(\cdot)$ and $V^*(\cdot)$ on $D[0, M]$ with local utility functions $U(\cdot; F)$ and $U^*(\cdot; F)$ with $U_1(\cdot; F)$, $U_1^*(\cdot; F) > 0$:*

(i) *for each $F(\cdot) \in D[0, M]$, $U^*(x; F) \equiv \lambda_F \cdot U(x; F) + G(x; F)$ for some $\lambda_F > 0$ and nonincreasing concave function $G(\cdot; F)$ satisfying $G_{11}(x; F) \leqslant G_1(y; F) \cdot U_{11}(x; F)/U_1(y; F)$ for all $x, y \in [0, M]$;*

[12] Although the definitions of "diversifiers" given in Tobin (1958) and Machina (1982a, 1982b) differ from this one due to the different contexts considered, they are similar in spirit. Except for the distinction between strict and weak quasiconcavity, this definition is equivalent to that of Dekel (1986).

(ii) $-U^*_{11}(x; F)/U^*_1(y; F) \geqslant -U_{11}(x; F)/U_1(y; F)$ *for all* $x, y \in [0, M]$ *and all* $F(\cdot) \in D[0, M]$; *and*

(iii) *if* π^* *and* π *solve* $V^*(F_{\tilde{x}-\pi^*\cdot\tilde{\eta}}) = V^*(F_{\tilde{x}+\tilde{\varepsilon}})$ *and* $V(F_{\tilde{x}-\pi\cdot\tilde{\eta}}) = V(F_{\tilde{x}+\tilde{\varepsilon}})$ *where* $\tilde{\eta} \geqslant 0$ *and* $E[\tilde{\varepsilon}|x] \equiv 0$, *then* $\pi^* \geqslant \pi$;

and if both individuals are diversifiers, these are equivalent to:

(iv) *if* $E[\tilde{z}|x] \geqslant 0$ *and* $\bar{\alpha}^*$ *and* $\bar{\alpha}$ *respectively maximize* $V^*(F_{\tilde{x}+\alpha\tilde{z}})$ *and* $V(F_{\tilde{x}+\alpha\tilde{z}})$, *then* $\bar{\alpha}^* \leqslant \bar{\alpha}$.

The proof is in the Appendix.

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Manuscript received April, 1986; final revision received December, 1986.

APPENDIX

Throughout the following we assume that the actual supports of all random variables such as $\tilde{x}+\tilde{\varepsilon}$, $\tilde{x}-\tilde{\pi}$, $\tilde{x}+\alpha\tilde{z}$, etc. lie in the interval $[0, M]$ so that we may assume that all integrals are taken over $(-\infty, +\infty)$ unless otherwise specified. As mentioned above, condition (i) of Theorem 2 and the original proof of its equivalence to (ii) is due to Eddie Dekel.

LEMMA: *If* $E[\tilde{\varepsilon}] = 0$, *then*

$$\int [U(x+\varepsilon) - U(x)] \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$
$$= \int_0^{\infty} \int_0^{\varepsilon} \int_0^{s} U_{11}(x+\omega) \cdot d\omega \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{-\infty}^{0} \int_{\varepsilon}^{0} \int_{s}^{0} U_{11}(x+\omega) \cdot d\omega \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon).$$

PROOF:

$$\int [U(x+\varepsilon) - U(x)] \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$
$$= \int_0^{\infty} [U(x+\varepsilon) - U(x)] \cdot dF_{\tilde{\varepsilon}}(\varepsilon) - \int_{-\infty}^{0} [U(x) - U(x+\varepsilon)] \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$
$$= \int_0^{\infty} \int_0^{\varepsilon} U_1(x+s) \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) - \int_{-\infty}^{0} \int_{\varepsilon}^{0} U_1(x+s) \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$
$$= \int_0^{\infty} \int_0^{\varepsilon} [U_1(x+s) - U_1(x)] \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{-\infty}^{0} \int_{\varepsilon}^{0} [U_1(x) - U_1(x+s)] \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$
$$= \int_0^{\infty} \int_0^{\varepsilon} \int_0^{s} U_{11}(x+\omega) \cdot d\omega \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{-\infty}^{0} \int_{\varepsilon}^{0} \int_{s}^{0} U_{11}(x+\omega) \cdot d\omega \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$

where the second-last equality above follows since $E[\tilde{\varepsilon}] = 0$ implies

$$0 = \int_0^{\infty} \varepsilon \cdot dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{-\infty}^{0} \varepsilon \cdot dF_{\tilde{\varepsilon}}(\varepsilon) = \int_0^{\infty} \int_0^{\varepsilon} 1 \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) - \int_{-\infty}^{0} \int_{\varepsilon}^{0} 1 \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon)$$

so that

$$0 = -\int_0^{\infty} \int_0^{\varepsilon} U_1(x) \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{-\infty}^{0} \int_{\varepsilon}^{0} U_1(x) \cdot ds \cdot dF_{\tilde{\varepsilon}}(\varepsilon). \qquad Q.E.D.$$

PROOF OF THEOREM 2: (i)→(ii): For each x, y, and $F(\cdot)$, (i) implies $G_{11}(x; F)\cdot U_1(y; F)\leq U_{11}(x; F)\cdot G_1(y; F)$, which implies $[G_{11}(x; F)+\lambda_F\cdot U_{11}(x; F)]\cdot U_1(y; F)\leq U_{11}(x; F)\cdot[G_1(y; F)+\lambda_F\cdot U_1(y; F)]$ and hence that $U^*_{11}(x; F)\cdot U_1(y; F)\leq U_{11}(x; F)\cdot U^*_1(y; F)$, which by positivity of $U_1(\cdot; F)$ and $U^*_1(\cdot; F)$ implies (ii).

(ii)→(i): For each $F(\cdot)$, define $\lambda_F=\max\{U^*_1(y; F)/U_1(y; F)|y\in[0, M]\}=U^*_1(\bar{y}; F)/U_1(\bar{y}; F)$ for some $\bar{y}\in[0, M]$ (so that $\lambda_F>0$) and $G(x; F)\equiv U^*(x; F)-\lambda_F\cdot U(x; F)$. This implies $G_1(x; F)\equiv U^*_1(x; F)-\lambda_F\cdot U_1(x; F)\leq 0$ and that $G_{11}(x; F)\equiv U^*_{11}(x; F)-\lambda_F\cdot U_{11}(x; F)\equiv U^*_{11}(x; F)-[U^*_1(\bar{y}; F)/U_1(\bar{y}; F)]\cdot U_{11}(x; F)$, which by (ii) is nonpositive. For each x, y, and $F(\cdot)$, (ii) implies $U^*_{11}(x; F)\cdot U_1(y; F)\leq U_{11}(x; F)\cdot U^*_1(y; F)$ which implies $[G_{11}(x; F)+\lambda_F\cdot U_{11}(x; F)]\cdot U_1(y; F)\leq U_{11}(x; F)\cdot[G_1(y; F)+\lambda_F\cdot U_1(y; F)]$ and hence that $G_{11}(x; F)\cdot U_1(y; F)\leq U_{11}(x; F)\cdot G_1(y; F)$, which by positivity of $U_1(\cdot; F)$ implies $G_{11}(x; F)\leq G_1(y; F)\cdot U_{11}(x; F)/U_1(y; F)$.

(ii)→(iii): Let $F(x, \varepsilon, \eta)$ be a joint distribution for which $E[\tilde{\varepsilon}|x]\equiv 0$ and $\tilde{\eta}\geq 0$, and let π solve $V(F_{\tilde{x}-\pi\cdot\tilde{\eta}})=V(F_{\tilde{x}+\tilde{\varepsilon}})$. Defining the joint distribution

$$\Phi(x, \varepsilon, \eta)\equiv\int_0^x F_{\tilde{\varepsilon}|x}(\varepsilon|\omega)\cdot F_{\tilde{\eta}|x}(\eta|\omega)\cdot dF_{\tilde{x}}(\omega),$$

it is straightforward to verify that $\Phi_{\tilde{x},\tilde{\varepsilon}}(\cdot,\cdot)=F_{\tilde{x},\tilde{\varepsilon}}(\cdot,\cdot)$ and $\Phi_{\tilde{x},\tilde{\eta}}(\cdot,\cdot)=F_{\tilde{x},\tilde{\eta}}(\cdot,\cdot)$, so that $\Phi_{\tilde{x}+\tilde{\varepsilon}}(\cdot)=F_{\tilde{x}+\tilde{\varepsilon}}(\cdot)$ and $\Phi_{\tilde{x}-\pi\cdot\tilde{\eta}}(\cdot)=F_{\tilde{x}-\pi\cdot\tilde{\eta}}(\cdot)$. Note also that under the distribution $\Phi(\cdot,\cdot,\cdot)$, $\tilde{\varepsilon}$ and $\tilde{\eta}$ are conditionally independent given x.[13]

For each $\tau\in[0, 1]$, define $\theta(\tau)$ as the solution[14] to

$$V((1-\tau)\cdot\Phi_{\tilde{x}+\tilde{\varepsilon}-\theta(\tau)\cdot\tilde{\eta}}+\tau\cdot\Phi_{\tilde{x}-\theta(\tau)\cdot\tilde{\eta}})=V(\Phi_{\tilde{x}-\pi\cdot\tilde{\eta}})=V(\Phi_{\tilde{x}+\tilde{\varepsilon}}),$$

and let $\Phi(\cdot;\tau)\equiv(1-\tau)\cdot\Phi_{\tilde{x}+\tilde{\varepsilon}-\theta(\tau)\cdot\tilde{\eta}}+\tau\cdot\Phi_{\tilde{x}-\theta(\tau)\cdot\tilde{\eta}}$. Note that $\theta(0)=0$, $\theta(1)=\pi$, $\Phi(\cdot;0)=\Phi_{\tilde{x}+\tilde{\varepsilon}}(\cdot)$ and $\Phi(\cdot;1)=\Phi_{\tilde{x}-\pi\cdot\tilde{\eta}}(\cdot)$, so that the set $\{\Phi(\cdot;\tau)|\tau\in[0, 1]\}$ forms a path from $\Phi_{\tilde{x}+\tilde{\varepsilon}}(\cdot)=F_{\tilde{x}+\tilde{\varepsilon}}(\cdot)$ to $\Phi_{\tilde{x}-\pi\cdot\tilde{\eta}}(\cdot)=F_{\tilde{x}-\pi\cdot\tilde{\eta}}(\cdot)$ in $D[0, M]$. From Machina (1982a, eq. 8) it follows that for each $\bar{\tau}\in[0, 1]$ we have

$$\begin{aligned}0&=\left.\frac{dV(\Phi(\cdot;\tau))}{d\tau}\right|_{\bar{\tau}}\\&=\frac{d}{d\tau}\Bigg[(1-\tau)\cdot\iiint U(x+\varepsilon-\theta(\tau)\cdot\eta;\Phi(\cdot;\bar{\tau}))\,d\Phi(x, \varepsilon, \eta)\\&\qquad+\tau\cdot\iiint U(x-\theta(\tau)\cdot\eta;\Phi(\cdot;\bar{\tau}))\,d\Phi(x, \varepsilon, \eta)\Bigg]\Bigg|_{\bar{\tau}}\\&=-\iiint[U(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))-U(x-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))]\,d\Phi(x, \varepsilon, \eta)\\&\qquad-\theta'(\bar{\tau})\cdot\iiint\eta\cdot[(1-\bar{\tau})\cdot U_1(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))\\&\qquad+\bar{\tau}\cdot U_1(x-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))]\,d\Phi(x, \varepsilon, \eta)\end{aligned}$$

so that $\theta'(\bar{\tau})$ is given by

$$-\frac{\iiint[U(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))-U(x-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))]\,d\Phi(x, \varepsilon, \eta)}{\iiint\eta\cdot[(1-\bar{\tau})\cdot U_1(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))+\bar{\tau}\cdot U_1(x-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))]\,d\Phi(x, \varepsilon, \eta)}.\tag{15}$$

By the definition of $\Phi(x, \varepsilon, \pi)$ we may write the numerator of (15) as

$$\begin{aligned}\iint\Bigg[\int&[U(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))\\&-U(x-\theta(\bar{\tau})\cdot\eta;\Phi(\cdot;\bar{\tau}))]\,dF_{\tilde{\varepsilon}|x}(\varepsilon|x)\Bigg]\,dF_{\tilde{x},\tilde{\eta}}(x, \eta)\end{aligned}$$

[13] When $F(x, \varepsilon, \eta)$ possesses a density function $f(x, \varepsilon, \eta)$, the density of $\Phi(x, \varepsilon, \eta)$ is $\phi(x, \varepsilon, \eta)\equiv f_{\tilde{\varepsilon}|x}(\varepsilon|x)\cdot f_{\tilde{\eta}|x}(\eta|x)\cdot f_{\tilde{x}}(x)$.

[14] Except for the trivial case where $\tilde{\eta}\equiv 0$, the existence and uniqueness of $\theta(\tau)$ for each τ follows from strict first order stochastic dominance preference for $V(\cdot)$, which in turn follows from the assumption that $U_1(\cdot; F)>0$ (see Machina (1982a)).

which from the Lemma is seen to equal

$$\iint \left[\int_0^\infty \int_0^\varepsilon \int_0^s U_{11}(x+\omega-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))\cdot d\omega\cdot ds\cdot dF_{\tilde{\varepsilon}|x}(\varepsilon|x) + \int_{-\infty}^0 \int_\varepsilon^0 \int_s^0 U_{11}(x+\omega-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))\cdot d\omega\cdot ds\cdot dF_{\tilde{\varepsilon}|x}(\varepsilon|x)\right] dF_{\tilde{x},\tilde{\eta}}(x,\eta).$$

Since the numerator and denominator of (15) can therefore be expressed as nonnegative weighted integrals of $U_{11}(\cdot;\Phi(\cdot;\bar{\tau}))$ and $U_1(\cdot;\Phi(\cdot;\bar{\tau}))$ respectively, an argument identical to that following equation (12) yields that (15) (i.e. $\theta'(\bar{\tau})$) is less than or equal to

$$-\frac{\iiint [U^*(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))-U^*(x-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))]\,d\Phi(x,\varepsilon,\eta)}{\iiint \eta\cdot[(1-\bar{\tau})\cdot U_1^*(x+\varepsilon-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))+\bar{\tau}\cdot U_1^*(x-\theta(\bar{\tau})\cdot\eta;\ \Phi(\cdot;\bar{\tau}))]\,d\Phi(x,\varepsilon,\eta)}.$$

But by an argument identical to the derivation of $dV(\Phi(\cdot;\tau))/d\tau|_{\bar{\tau}}$ above this implies that $dV^*(\Phi(\cdot;\tau))/d\tau|_{\bar{\tau}}\geq 0$ for all $\bar{\tau}\in[0,1]$, so that $V^*(F_{\tilde{x}-\pi\cdot\tilde{\eta}})\geq V^*(F_{\tilde{x}+\tilde{\varepsilon}})=V^*(F_{\tilde{x}-\pi^*\cdot\tilde{\eta}})$, and hence that $\pi^*\geq\pi$.

(iii)$\to$(ii): Assume that $-U_{11}^*(x_0;F_0)/U_1^*(y_0;F_0)<-U_{11}(x_0;F_0)/U_1(y_0;F_0)$ for some $x_0,y_0\in[0,M]$ and $F_0(\cdot)\in D[0,M]$ (by smoothness we may assume $x_0\neq y_0$ and that $p_0=\text{prob}\,(x_0)$ and $q_0=\text{prob}\,(y_0)$ are both positive). Define $F(x,\nu,\eta)$ such that $F_{\tilde{x}}(\cdot)=F_0(\cdot)$, $\tilde{\eta}=1$ if $\tilde{x}=y_0$ ($\tilde{\eta}=0$ otherwise), and $\tilde{\nu}$ is a 50:50 chance of ±1 if $\tilde{x}=x_0$ ($\tilde{\nu}=0$ otherwise). Defining $\pi^*(t)$ and $\pi(t)$ as the solutions to $V^*(F_{\tilde{x}-\pi^*(t)\cdot\tilde{\eta}})=V^*(F_{\tilde{x}+\sqrt{t}\cdot\tilde{\nu}})$ and $V(F_{\tilde{x}-\pi(t)\cdot\tilde{\eta}})=V(F_{\tilde{x}+\sqrt{t}\cdot\tilde{\nu}})$ for each t, we have from Machina (1982a, eq. 8) and equation (11) that $d\pi^*(t)/dt|_{t=0}=-\frac{1}{2}\cdot U_{11}^*(x_0)\cdot p_0/U_1^*(y_0)\cdot q_0<-\frac{1}{2}\cdot U_{11}(x_0)\cdot p_0/U_1(y_0)\cdot q_0=d\pi(t)/dt|_{t=0}$, so that $\pi^*(t_0)<\pi(t_0)$ for some small positive t_0. Defining $\tilde{\varepsilon}=\sqrt{t_0}\cdot\tilde{\nu}$ yields a contradiction of (iii).

(i)$\to$(iv): Let $F(\cdot,\cdot)$ be the joint distribution of $(\tilde{x},\tilde{z})$. Defining $F(\cdot;\alpha)\equiv F_{\tilde{x}+\alpha\tilde{z}}(\cdot)$, we have from Machina (1982a, eq. 8) that $\bar{\alpha}$ satisfies

$$\begin{aligned}0&=\frac{d}{d\alpha}[V(F(\cdot;\alpha))]\bigg|_{\bar{\alpha}}=\frac{d}{d\alpha}\left[\int U(\omega;F(\cdot;\bar{\alpha}))\,dF_{\tilde{x}+\alpha\tilde{z}}(\omega)\right]\bigg|_{\bar{\alpha}}\\&=\frac{d}{d\alpha}\left[\iint U(x+\alpha z;F(\cdot;\bar{\alpha}))\,dF(x,z)\right]\bigg|_{\bar{\alpha}}\\&=\iint z\cdot U_1(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))\,dF(x,z).\end{aligned}$$

A similar derivation yields that

$$\begin{aligned}\frac{d}{d\alpha}[V(F(\cdot;\alpha))]\bigg|_{\alpha=0}&=\iint z\cdot U_1(x;F_{\tilde{x}})\,dF(x,z)\\&=\int U_1(x;F_{\tilde{x}})\left[\int z\cdot dF_{\tilde{z}|x}(z|x)\right]dF_{\tilde{x}}(x)\geq 0\end{aligned}$$

which implies $\bar{\alpha}\geq 0$. We also have,

$$\begin{aligned}\frac{d}{d\alpha}[V^*(F(\cdot;\alpha))]\bigg|_{\bar{\alpha}}&=\iint z\cdot U_1^*(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))\,dF(x,z)\\&=\iint z\cdot[\lambda_{F(\cdot;\bar{\alpha})}\cdot U_1(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))\\&\qquad+G_1(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))]\,dF(x,z)\end{aligned}$$

which from above is seen to equal

$$\begin{aligned}&\iint z\cdot G_1(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))\,dF(x,z)\\&\quad=\int\left[\int z\cdot G_1(x+\bar{\alpha}z;F(\cdot;\bar{\alpha}))\,dF_{\tilde{z}|x}(z|x)\right]dF_{\tilde{x}}(x).\end{aligned}$$

Since $G_1, G_{11} \leqslant 0$, $\bar{\alpha} \geqslant 0$ and $E[\tilde{z}|x] \geqslant 0$, the bracketed term will be nonpositive. Since this implies that $dV^*(F(\cdot; \alpha))/d\alpha|_{\bar{\alpha}} \leqslant 0$ and $V^*(\cdot)$ is a diversifier, it follows that $\bar{\alpha}^* \leqslant \bar{\alpha}$.

(iv) $\rightarrow$ (ii): Assume $-U_{11}^*(x_0; F_0)/U_1^*(y_0; F_0) < \lambda < -U_{11}(x_0; F_0)/U_1(y_0; F_0)$ for some x_0, y_0, λ, and $F_0(\cdot)$. By smoothness we may assume that x_0, y_0, λ, and $F_0(\cdot)$ are such that $x_0 \neq y_0$, prob $(x_0) =$ prob $(y_0) = 0$, and that there exist small positive γ and p_0 such that prob $(x_0 + \gamma) =$ prob $(x_0 - \gamma) =$ prob $(y_0 + \lambda\gamma) = p_0$ and such that

$$\frac{U_1^*(x_0+\gamma; F_0) - U_1^*(x_0-\gamma; F_0)}{U_1^*(y_0+\lambda\gamma; F_0)} > -\lambda > \frac{U_1(x_0+\gamma; F_0) - U_1(x_0-\gamma; F_0)}{U_1(y_0+\lambda\gamma; F_0)}.$$

Define $(\tilde{x}, \tilde{z})$ such that $F_{\tilde{x}}(\cdot) = F_0(\cdot) + 2p_0 \cdot \delta_{x_0}(\cdot) + p_0 \cdot \delta_{y_0}(\cdot) - p_0 \cdot \delta_{(x_0+\gamma)}(\cdot) - p_0 \cdot \delta_{(x_0-\gamma)}(\cdot) - p_0 \cdot \delta_{y_0+\lambda\gamma}(\cdot)$, where $\delta_c(\cdot)$ denotes that distribution which assigns unit probability to c, and $\tilde{z}$ is a 50:50 chance of ± 1 if $\tilde{x} = x_0$, $\tilde{z} = \lambda$ if $\tilde{x} = y_0$, and $\tilde{z} = 0$ otherwise. We therefore have that $F_{\tilde{x}+\alpha\tilde{z}}(\cdot)$ equals $F_0(\cdot)$ when $\alpha = \gamma$. It follows that $dV(F_{\tilde{x}+\alpha\tilde{z}})/d\alpha|_{\alpha=\gamma} = p_0 \cdot U_1(x_0+\gamma; F_0) - p_0 \cdot U_1(x_0-\gamma; F_0) + \lambda \cdot p_0 \cdot U_1(y_0+\lambda\gamma; F_0)$, which by the previous inequality must be negative, so that the optimal value $\bar{\alpha}$ is less than γ. A similar argument establishes that the optimal value $\bar{\alpha}^*$ is greater than γ, which is a contradiction. *Q.E.D.*

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