

Alternative Approaches to Comparative n th-Degree Risk Aversion

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Abstract. This paper extends the three main approaches to comparative risk aversion—the risk premium approach and the probability premium approach of Pratt (1964) [Risk aversion in the small and in the large. *Econometrica* 32(1-2):122–136] and the comparative statics approach of Jindapon and Neilson (2007) [Higher-order generalizations of Arrow–Pratt and Ross risk aversion: A comparative statics approach. *J. Econom. Theory* 136(1):719–728]—to study comparative n th-degree risk aversion. These extensions can accommodate trading off an n th-degree risk increase and an m th-degree risk increase for any m , such that $1 \leq m < n$. It goes on to show that, in the expected utility framework, all of these general notions of comparative n th-degree risk aversion are equivalent and can be characterized by the concept of (n/m) th-degree Ross more risk aversion of Liu and Meyer (2013) [Substituting one risk increase for another: A method for measuring risk aversion. *J. Econom. Theory* 148(6):2706–2718].

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1. Introduction

For more than half a century now, economists have used the Arrow–Pratt measure of risk aversion to compare how risk averse two individuals are. There are good reasons why. For one thing, the mathematical characterization is very simple: based on a ratio of the first two derivatives of the utility function. Second and perhaps more importantly, the comparison based on the Arrow–Pratt measure was accompanied by some mathematically equivalent behavioral conditions. Specifically, being everywhere higher on the Arrow–Pratt measure is equivalent to having a larger risk premium or having a larger probability premium for an actuarially neutral risk (Pratt 1964). These behavioral or choice-based measures of risk aversion (i.e., the risk premium and the probability premium) have the advantage of being readily computed and compared in experiments investigating the factors that affect the strength of risk aversion.¹

Recent experimental studies have shown, in various contexts, a salient aversion to risk increases of third and even higher degrees.² Moreover, third-degree risk aversion (i.e., downside risk aversion or prudence) or even higher-degree risk aversion has been shown to play critical roles in some important models of decision making under risk. One example is the self-protection decision. Although a risk-averse individual does not necessarily invest more in self-protection than a risk-neutral individual, a downside risk-averse individual tends to invest less in self-protection than a downside risk-neutral individual (Chiu 2005, Eeckhoudt and

Gollier 2005, Menegatti 2009, Crainich et al. 2016, Denuit et al. 2016, Peter 2017). Another example is the precautionary saving/effort decision. It has been shown that, as future income undergoes an n th-degree risk increase, precautionary saving increases if and only if the utility function displays $(n + 1)$ th-degree risk aversion (Leland 1968, Sandmo 1970, Dreze and Modigliani 1972, Kimball 1990, Eeckhoudt and Schlesinger 2008, Eeckhoudt et al. 2012, Liu 2014, Wang et al. 2015, Nocetti 2016).

Along with these interests in higher-degree risk aversion, a question arises as to how to compare two individuals' relative strength of higher-degree risk aversion. The risk premium approach of Pratt (1964) to comparative risk aversion has been generalized to deal with random initial wealth and comparative higher-degree risk aversion.³ By comparison, his probability premium approach to comparative risk aversion, although extensively used in experiments investigating the strength of second-degree risk aversion, has not played an important role in the study of comparative higher-degree risk aversion.

More recently, Jindapon and Neilson (2007) propose a new approach to comparative n th-degree risk aversion that is based on a comparative statics analysis. The intuition for this approach is the following. Given a continuous opportunity to reduce the n th-degree risk in one's wealth by incurring a monetary cost, a "more n th-degree risk averse" individual would choose to incur a larger monetary cost to reduce the n th-degree risk further. They show that, in the expected utility framework,

an individual would always be willing to incur a larger monetary cost to reduce the n th-degree risk in wealth than another individual if and only if the former is n th-degree Ross more risk averse than the latter.

As Liu and Meyer (2013) argue, however, all of these existing approaches to comparative n th-degree risk aversion essentially quantify n th-degree risk aversion by the willingness to trade a first-degree risk increase with an n th-degree risk increase. They propose a notion of “risk trade-off”—the ratio of the reduction in expected utility caused by an n th-degree risk increase to that caused by an m th-degree risk increase—to quantify n th-degree risk aversion through the willingness to trade an m th-degree risk increase with an n th-degree risk increase for any m , such that $1 \leq m < n$. They further show that an individual always has a larger trade-off between an n th-degree risk increase and an m th-degree risk increase than another if and only if the former is (n/m) th-degree Ross more risk averse than the latter.

Nevertheless, the “risk trade-off” approach to comparative risk aversion of Liu and Meyer (2013) has an important limitation. Their “risk trade-off” or “rate of substitution” is defined in the expected utility framework, and actually calculating it requires knowing the utility function of the decision maker, which greatly hinders the experimental implementation of their approach. In contrast, the risk premium and probability premium by Pratt (1964) and the optimal monetary investment in reducing the n th-degree risk of wealth by Jindapon and Neilson (2007) are all choice based and can be assessed in appropriately designed experiments without requiring knowledge of a utility function.

This paper extends the three main behavioral approaches to comparative risk aversion—the risk premium approach and the probability premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007)—to study comparative n th-degree risk aversion, accommodating trading an m th-degree risk increase with an n th-degree risk increase for any m , such that $1 \leq m < n$. We show that, within the expected utility framework, behavior in all of these approaches is governed by the same mathematical condition based on the ratio of the n th and m th derivatives of the utility functions.⁴ Accordingly, all of these behavioral conditions are equivalent, and decision analysts can treat all of them as appropriate avenues for measuring and interpreting higher-order comparative risk attitudes.

First, we consider a situation where the individual compares random initial wealth with a binary compound lottery, where the “good” state has less m th-degree risk than initial wealth and the “bad” state has higher n th-degree risk than initial wealth, with $1 \leq m < n$.⁵ Generalizing the probability premium of Pratt (1964), we look for the probability of the good state that makes the individual indifferent between initial wealth and the binary compound lottery. The relevant behavioral condition

for one individual to be more n th-degree risk averse than another is that the former requires a higher probability on the good state than the latter for every initial wealth, every n th-degree risk increase, and every m th-degree risk decrease. We start with this generalized probability premium approach, because it seems to be the most intuitive one as well as the most convenient one to be implemented in experiments.

Second, we propose a notion of the path-dependent m th-degree risk premium for an n th-degree risk increase and interpret the existing risk premium concepts as the first-degree risk premium along some special paths. The relevant behavioral condition for one individual to be more n th-degree risk averse than another is that the former has a larger path-dependent m th-degree risk premium than the latter for every (random) initial wealth, every n th-degree risk increase, and every possible path of m th-degree increasing risk.

Third, we study a decision problem in which an individual faces an indexed path of random variables, and movements along the path involve precisely defined reductions in n th-degree risk and increases in m th-degree risk. This formulation has as special cases both the comparative statics problem analyzed in Jindapon and Neilson (2007) and the portfolio choice problem analyzed in Pratt (1964), Ross (1981), and Machina and Neilson (1987). The relevant behavioral condition for one individual to be more n th-degree risk averse than another is that the former always chooses a random variable farther along the path than the latter. More simply, the n th-degree more risk-averse individual chooses a random variable with less n th-degree risk but at the cost of more m th-degree risk.

Importantly and unlike the derivative conditions, these general behavioral notions of comparative n th-degree risk aversion are all model-free; in particular, they do not depend on the expected utility framework to be well defined. As a result, they can be readily computed/tested based on decisions solicited in appropriately designed experiments. Nevertheless, the paper goes on to show that, when the expected utility framework is assumed, all of these general behavioral notions of comparative n th-degree risk aversion are equivalent and can be characterized by the (n/m) th-degree Ross more risk aversion of Liu and Meyer (2013). Not only does this equivalence provide a unified choice-based justification for the concept of (n/m) th-degree Ross more risk aversion, it also provides practitioners with a variety of methods for assessing and explaining comparative higher-degree risk aversion.

The paper is organized as follows. Section 2 reviews notions of n th-degree increasing risk, n th-degree risk aversion, and (n/m) th-degree Ross more risk aversion. Section 3 presents three behavioral (i.e., choice-based) conditions comparing the n th-degree risk aversion of two individuals as generalizations of the three main approaches to comparative risk aversion—the probability

premium approach, the risk premium approach, and the comparative statics approach. The theorems in this section establish that, in the framework of expected utility, all of these three behavioral conditions are equivalent and characterized by the (n/m) th-degree Ross more risk-averse condition. Section 4 offers some conclusions.

2. n th-Degree Increases in Risk, n th-Degree Risk Aversion, and (n/m) th-Degree Ross More Risk Aversion

Let $F(x)$ and $G(x)$ represent the cumulative distribution functions (CDFs) of two random variables with supports that are contained in a finite interval denoted $[a, b]$ with no probability mass at point a . This implies that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Letting $F^{[1]}(x)$ denote $F(x)$, higher-order cumulative functions are defined according to $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy$, $k = 2, 3, \dots$. Similar notation applies to $G(x)$ and other CDFs.

For any integer $n \geq 1$, Ekern (1980) gives the following definition.

Definition 1. $G(x)$ has more n th-degree risk (or is more n th-degree risky) than $F(x)$ if

$$\begin{aligned} G^{[k]}(b) &= F^{[k]}(b) \quad \text{for } k = 1, 2, \dots, n \text{ and} & (1) \\ G^{[n]}(x) &\geq F^{[n]}(x) \quad \text{for all } x \text{ in } [a, b], \text{ with " } > \text{ "} \\ &\quad \text{holding for some } x \text{ in } (a, b). & (2) \end{aligned}$$

Condition (1) guarantees that the first $n - 1$ moments are held constant across the two distributions, and Conditions (1) and (2) together imply that $F(x)$ dominates $G(x)$ in n th-degree stochastic dominance. Thus, the n th-degree risk increase is a special case of n th-degree stochastic dominance, in which the first $n - 1$ moments are kept the same. This general definition of n th-degree risk increases has many well-known notions of stochastic changes as special cases. An increase in first-degree risk is a first-order stochastically dominated shift, which visually entails a leftward shift in probability mass. It implies (but is not equivalent to) a reduction in the mean. An increase in second-degree risk is simply the familiar risk increase of Rothschild and Stiglitz (1970), which corresponds to a sequence of mean-preserving spreads. It implies (but is not equivalent to) an increase in the variance. Similarly, an increase in third-degree risk holds the first two moments constant and shifts risk from high-wealth levels to low-wealth levels, which is the downside risk increase of Menezes et al. (1980). It implies (but is not equivalent to) a reduction in rightward (positive) skewness.

The fact that an n th-degree risk increase requires that the first $n - 1$ moments remain constant places a restriction on the starting distribution $F(x)$. In particular, when $F(x)$ is degenerate, placing all of its probability mass on a single outcome x_0 , only first-degree and second-degree risk increases are possible.

A first-degree risk increase would entail a first-order stochastically dominated shift as usual, and a second-degree one would involve a mean-preserving spread. A third-degree or higher risk increase would require that the variance of the new distribution be the same as that of the original distribution, and a degenerate distribution has zero variance. Consequently, third-degree or higher risk increases are only well defined when the starting distribution is nondegenerate.

Ekern (1980) also provides a definition of n th-degree risk aversion when the preferences have an expected utility representation. For any utility function $u(x): [a, b] \rightarrow \mathbb{R}$, assume that $u(x)$ has all possible derivatives and that they are continuous. Denote by $u^{(k)}(x)$ the k th derivative of $u(x)$, $k = 1, 2, 3, \dots$

Definition 2. Decision-maker $u(x)$ is n th-degree risk averse if $(-1)^{n+1}u^{(n)}(x) > 0$ for all x in $[a, b]$.

We use u and v to refer to decision makers and also use utility functions $u(x)$ and $v(x)$ to refer to decision makers with the corresponding utility functions when the expected utility framework is assumed. Note that $u(x)$ is said to be *weakly* n th-degree risk averse when the strict inequality in Definition 2 is replaced with a weak one. First-degree risk aversion corresponds to an everywhere increasing utility function, and the usual second-degree risk aversion corresponds to a concave utility function. If an individual exhibits all possible degrees of risk aversion, his utility function will have derivatives that alternate in sign, beginning with a positive first derivative.⁶

The relationship between the two concepts in Definitions 1 and 2 is given in Lemma 1 below, which is proved by Ekern (1980).⁷

Lemma 1. $G(x)$ has more n th-degree risk than $F(x)$ if and only if every n th-degree risk-averse decision-maker $u(x)$ prefers $F(x)$ to $G(x)$.

This result shows that n th-degree increases in risk are precisely the distribution changes that every n th-degree risk-averse individual dislikes.

Another definition that is necessary for the analysis in this paper is (n/m) th-degree Ross more risk aversion, which was first described by Liu and Meyer (2013). Assume that m and n are two positive integers, such that $1 \leq m < n$, and let the two utility functions $u(x)$ and $v(x)$ each be both n th-degree and m th-degree risk averse on $[a, b]$. The following definition of (n/m) th-degree Ross more risk aversion is from Liu and Meyer (2013).

Definition 3. $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$ on $[a, b]$ if

$$\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(y)} \geq \frac{(-1)^{n+1}v^{(n)}(x)}{(-1)^{m+1}v^{(m)}(y)} \quad \text{for all } x, y \in [a, b], \tag{3}$$

or equivalently, if there exists $\lambda > 0$, such that $\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)}$ for all $x, y \in [a, b]$.

Definition 3 includes many existing notions of one utility function being more risk averse than another as special cases. For $n = 2, m = 1$, and $y = x$, Condition (3) reduces to the familiar Arrow–Pratt more risk-averse condition: $-\frac{u''(x)}{u'(x)} \geq -\frac{v''(x)}{v'(x)}$ for all $x \in [a, b]$. As Ross (1981) points out, the behavioral conditions related to this characterization must have nonstochastic initial wealth, and the stronger condition $-\frac{u''(x)}{u'(y)} \geq -\frac{v''(x)}{v'(y)}$ for all $x, y \in [a, b]$ —which is referred to in the literature as Ross more risk aversion—allows for random initial wealth. For $m = 1$, Definition 3 reduces to Ross more risk aversion when $n = 2$, Ross more downside risk aversion when $n = 3$ (Modica and Scarsini 2005), and Ross n th-degree more risk aversion for a general $n \geq 2$ (Jindapon and Neilson 2007, Li 2009, Denuit and Eeckhoudt 2010).

The following lemmas regarding the (n/m) th-degree Ross more risk-averse condition will be used in proving the main results in the paper. Specifically, Lemma 2 is useful when using (n/m) th-degree Ross more risk aversion as a sufficient condition, and Lemma 3 is useful when showing (n/m) th-degree Ross more risk aversion as a necessary condition.⁸ A proof of Lemma 2 is given in Liu and Meyer (2013),⁹ and a proof of Lemma 3 is provided in the appendix.

Lemma 2. $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$ on $[a, b]$ if and only if there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all x in $[a, b]$, such that $u(x) \equiv \lambda v(x) + \phi(x)$.

Lemma 3. If $u(x)$ is not (n/m) th-degree Ross more risk averse than $v(x)$ on $[a, b]$, then there exist $\mu > 0, [a_1, b_1] \subset (a, b)$, and $[a_2, b_2] \subset (a, b)$, such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} (-1)^{n+1}\phi^{(n)}(x) &< 0 && \text{for all } x \in [a_1, b_1] \\ (-1)^{m+1}\phi^{(m)}(x) &> 0 && \text{for all } x \in [a_2, b_2]. \end{aligned}$$

Other than its usefulness for proofs, Lemma 2 provides a way to construct utility functions that are (n/m) th-degree Ross more risk averse than a given one, which might be useful for applied work. In addition, it is clear from Lemma 2 that assuming two utility functions to be ranked according to the (n/m) th-degree Ross more risk-averse relation is by no means a vacuous assumption.

3. Alternative Approaches to Comparative n -Degree Risk Aversion

3.1. The Probability Premium Approach

Pratt (1964) proposes to use the probability premium as a measure of (global) risk aversion. Pratt (1964) defines

the probability premium q according to the indifference condition

$$w \sim \begin{cases} w + \varepsilon & \text{with probability } \frac{1}{2} + q \\ w - \varepsilon & \text{with probability } \frac{1}{2} - q, \end{cases} \quad (4)$$

where w is the nonrandom initial wealth and $\varepsilon > 0$ is a constant. Pratt (1964) further shows that, in the expected utility framework, an individual $u(x)$ always has a larger probability premium than another individual $v(x)$ —for all w and ε —if and only if the former is Arrow–Pratt more risk averse than the latter.

Unlike the risk premium approach that is discussed in the next subsection, the probability premium approach to comparative risk aversion has not played an important role in understanding comparative higher-degree risk aversion, although probability-type measures of second-degree risk aversion have been extensively used in experiments.¹⁰ The reason for this is probably that the probability premium was not used by Ross (1981) in generalizing the analysis of Pratt (1964) from nonstochastic initial wealth to random initial wealth, and one has to work with random initial wealth when studying third-degree or even higher-degree risk increases.

We propose below a general formulation for using the probability premium to measure n th-degree risk aversion. Suppose that \tilde{w} is initial wealth, \tilde{y} is an n th-degree risk increase from \tilde{w} , and \tilde{z} is an m th-degree risk decrease from \tilde{w} . For an individual who is both n th-degree and m th-degree risk averse, $\tilde{z} > \tilde{w} > \tilde{y}$, where “ $>$ ” denotes the strict preference relationship. Consider a two-state compound lottery:

$$\begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1 - p. \end{cases}$$

As p increases continuously from zero to one, the above lottery goes from being dominated by \tilde{w} to being preferred to \tilde{w} . Assuming continuity of preferences with respect to p , meaning that a very small change in p would not reverse a preference relation between the two lotteries compared, there exists a p , such that

$$\tilde{w} \sim \begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1 - p. \end{cases} \quad (5)$$

Formally, the m th-degree probability premium for an n th-degree risk increase is defined below.

Definition 4. Suppose that \tilde{w} is the random initial wealth, \tilde{y} is an n th-degree risk increase from \tilde{w} , and \tilde{z} is an m th-degree risk decrease from \tilde{w} . The m th-degree probability premium for the n th-degree risk increase is the scalar p satisfying the indifference Condition (5).

Note that, for $n = 2$ and $m = 1$, the m th-degree probability premium for the n th-degree risk increase includes the probability premium of Pratt (1964) as a special case. To see this, let $\tilde{w} = w$, $\tilde{z} = w + \varepsilon$, and \tilde{y} have two outcomes, $w + \varepsilon$ and $w - \varepsilon$, with equal probability $1/2$. Then, (5) becomes

$$w \sim \begin{cases} w + \varepsilon & \text{with probability } \frac{1}{2} + \frac{p}{2} \\ w - \varepsilon & \text{with probability } \frac{1}{2} - \frac{p}{2}, \end{cases}$$

which is exactly the indifference Condition (4) after relabeling $\frac{p}{2}$ as q .

It is straightforward to see that, if the individual is both m th-degree and n th-degree risk averse, then any m th-degree probability premium for an n th-degree risk increase is in $(0, 1)$. Now consider two individuals, u and v , with different risk preferences. Given \tilde{w} , \tilde{y} , and \tilde{z} , if p_u and p_v satisfy (5) for \sim_u and \sim_v , respectively, and $p_u > p_v$, then this means that, compared with v , individual u requires a larger probability on the favorable state—in which an m th-degree risk decrease materializes—for the two-state compound lottery to be indifferent to the status quo.

If $p_u \geq p_v$ for all \tilde{w} , \tilde{y} , and \tilde{z} , then u can be regarded as being more n th-degree risk averse than v when the necessary compensation to offset an n th-degree risk increase takes the form of an m th-degree risk decrease. The following theorem shows that, in the framework of expected utility, the condition $p_u \geq p_v$ is characterized by (n/m) th-degree Ross more risk aversion. The proof of the theorem is in the appendix.

Theorem 1. *Suppose that two expected utility maximizers $u(x)$ and $v(x)$ are each both m th-degree risk averse and n th-degree risk averse everywhere. The m th-degree probability premia satisfy $p_u \geq p_v$ for every \tilde{w} , \tilde{y} , and \tilde{z} , such that \tilde{y} is n th-degree more risky than \tilde{w} and \tilde{z} is m th-degree less risky than \tilde{w} if and only if $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$.*

Theorem 1 provides a straightforward way for understanding what (n/m) th-degree Ross more risk averse means. Individuals have initial random wealth given by \tilde{w} and consider replacing it with a binary compound lottery that pays random variable \tilde{z} in the good state and random variable \tilde{y} in the bad state. What makes the bad state bad is that \tilde{y} is n th-degree riskier than \tilde{w} , and what makes the good state good is that \tilde{z} is m th-degree less risky than \tilde{w} . Choosing to move away from the status quo then involves trading off the n th-degree risk increase against the m th-degree risk reduction. The individual who is (n/m) th-degree Ross more risk averse requires a larger probability on the m th-degree risk reduction to keep him indifferent between the binary compound lottery and the status quo, which means, for him, the n th-degree risk increase weighs

relatively more heavily in his decision than the m th-degree risk reduction does compared with the other individual.

3.2. The Risk Premium Approach

The best-known approach to comparative risk aversion involves the risk premium. In the original Arrow–Pratt analysis, the decision maker has nonstochastic initial wealth w and faces an additive mean zero risk $\tilde{\varepsilon}$. The risk premium π is the payment that satisfies the indifference condition $w - \pi \sim w + \tilde{\varepsilon}$. Ross (1981) extends the Arrow–Pratt analysis to random starting wealth levels and defines the risk premium π according to

$$\tilde{w} - \pi \sim \tilde{y}, \tag{6}$$

where \tilde{w} is the random initial wealth and \tilde{y} is a Rothschild–Stiglitz risk increase from \tilde{w} . In the expected utility framework, Ross (1981) shows that an individual always has a larger risk premium than another—for all \tilde{w} and \tilde{y} —if and only if the former is Ross more risk averse than the latter.¹¹

Machina and Neilson (1987) extend Ross (1981) by defining a random risk premium. More precisely, suppose that \tilde{w} is the initial wealth, \tilde{y} is a Rothschild–Stiglitz risk increase from \tilde{w} , and $\tilde{\eta}$ is a nonnegative random variable. The random risk premium π is a scalar satisfying the indifference condition

$$\tilde{w} - \pi\tilde{\eta} \sim \tilde{y}. \tag{7}$$

Machina and Neilson (1987) further show that, in the expected utility framework, an individual always has a larger random risk premium than another—for all \tilde{w} , \tilde{y} , and $\tilde{\eta}$ —if and only if the former is Ross more risk averse than the latter.

Note that the left-hand side of (6) or (7) ($\tilde{w} - \pi$ or $\tilde{w} - \pi\tilde{\eta}$) is a first-degree risk increase from \tilde{w} when $\pi > 0$, and the right-hand side (\tilde{y}) is a second-degree risk increase from \tilde{w} . Therefore, the risk premium Conditions (6) and (7) involve trading off a first-degree risk increase against a second-degree one along their respective “path” of first-degree risk increases. Take (7), for example. The set $\{\tilde{w} - \pi\tilde{\eta}\}_{\pi \in \mathbb{R}^+}$, where $\mathbb{R}^+ = [0, \infty)$, constitutes a continuous parameterized path indexed by the scalar π .¹² Along this path, higher values of π correspond to increases in first-degree risk, and identifying π in Expression (7) is the same as finding the random variable on the path that is indifferent to \tilde{y} . A random variable farther along the path involves more first-degree risk and therefore, a larger random risk premium, and consequently, an individual who moves farther along the path to reach indifference has a higher risk premium than one who does not move as far.

We can use this continuous path idea to formulate a general definition of the *path-dependent m th-degree risk premium* for an n th-degree risk increase,

where $1 \leq m < n$, and use it to measure an individual's n th-degree risk aversion in terms of an m th-degree risk increase. Let \tilde{w} be the random initial wealth and \tilde{y} be an n th-degree risk increase from \tilde{w} , and let $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$ denote a continuous path of random variables parameterized by $\pi \in [0, B] \subset \mathbb{R}^+$, such that $\tilde{x}(0) = \tilde{w}$; for every $\pi' > \pi \geq 0$, the random variable $\tilde{x}(\pi')$ has more m th-degree risk than $\tilde{x}(\pi)$ does, where $1 \leq m < n$. We refer to $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$ as a *path of m th-degree increasing risk from \tilde{w}* .

Definition 5. Suppose that \tilde{w} is the random initial wealth, \tilde{y} is an n th-degree risk increase from \tilde{w} , and $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$ is a path of m th-degree increasing risk from \tilde{w} with $\tilde{x}(B) < \tilde{y}$. The path-dependent m th-degree risk premium is the scalar π satisfying the indifference condition

$$\tilde{x}(\pi) \sim \tilde{y}. \tag{8}$$

Obviously, $\{\tilde{w} - \pi\}_{\pi \in \mathbb{R}^+}$ and $\{\tilde{w} - \pi\tilde{\eta}\}_{\pi \in \mathbb{R}^+}$ are examples of paths of first-degree increasing risk from \tilde{w} . The following examples are some paths of m th-degree increasing risk from \tilde{w} for $m \geq 2$. First, $\{\tilde{x}(\pi) = \tilde{w} + \pi\tilde{\varepsilon}\}_{\pi \in \mathbb{R}^+}$, where $\tilde{\varepsilon}$ is a mean zero nondegenerate risk that is independent of \tilde{w} , is a path of second-degree increasing risk from \tilde{w} . Second, suppose that \tilde{z} (with CDF $H(x)$) has more m th-degree risk than \tilde{w} (with CDF $F(x)$). Then, $\{\tilde{x}(\pi)\}_{\pi \in [0, 1]}$ is a path of m th-degree increasing risk from \tilde{w} if $\tilde{x}(\pi)$ has a CDF of $\pi H(x) + (1 - \pi)F(x)$. In fact, assuming the expected utility framework and representing the preferences by utility function $u(x)$, the path-dependent m th-degree risk premium for an n th-degree risk increase from \tilde{w} to \tilde{y} along this path is given by $\pi Eu(\tilde{z}) + (1 - \pi)Eu(\tilde{w}) = Eu(\tilde{y})$ or

$$\pi = \frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})}. \tag{9}$$

Note that the ratio in (9) is the “rate of substitution” or “risk trade-off” between an n th-degree risk increase and an m th-degree risk increase defined in Liu and Meyer (2013). Therefore, their rate of substitution is the path-dependent m th-degree risk premium for an n th-degree risk increase along a special path of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0, 1]}$, as discussed above.

It is straightforward to see that, if the individual is both m th-degree and n th-degree risk averse, any path-dependent m th-degree risk premium for an n th-degree risk increase must be positive. Now consider two individuals, u and v , with different risk preferences. Take as given \tilde{w} , \tilde{y} , and a path of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$. If π_u and π_v satisfy $\tilde{x}(\pi_u) \sim_u \tilde{y}$ and $\tilde{x}(\pi_v) \sim_v \tilde{y}$, respectively, and $\pi_u > \pi_v$, then this means that, compared with v , individual u must move farther along the path of m th-degree increasing risk from \tilde{w} before offsetting the disutility caused by the n th-degree increase in risk entailed in \tilde{y} . More to the point and

much like the original Arrow–Pratt case, individual u is willing to accept a larger m th-degree risk increase to avoid an n th-degree risk increase than individual v .

If $\pi_u \geq \pi_v$ for all \tilde{w} , \tilde{y} , and paths of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$, then u can be regarded as being more n th-degree risk averse than v when the willingness to pay for avoiding the n th-degree risk increase takes the form of an m th-degree risk increase. The following theorem provides a utility function-based characterization of the condition $\pi_u \geq \pi_v$ in the tradition of Pratt (1964), when the preferences of both u and v satisfy the axioms of expected utility and are represented by utility functions $u(x)$ and $v(x)$, respectively. The proof of the theorem is in the appendix.

Theorem 2. Suppose that two expected utility maximizers $u(x)$ and $v(x)$ are both m th-degree risk averse and n th-degree risk averse everywhere. The path-dependent m th-degree risk premia satisfy $\pi_u \geq \pi_v$ for every \tilde{w} , every \tilde{y} that is n th-degree riskier than \tilde{w} , and every path of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$, if and only if $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$.

The condition that both individuals are everywhere m th-degree risk averse plays the same role that increasing utility functions plays in the standard Arrow–Pratt characterization of second-degree comparative risk aversion. There, the increasing utility functions imply that the individual dislikes increases in the risk premium, and here, the m th-degree risk aversion implies that the individual dislikes movements farther along the path of m th-degree increasing risk from \tilde{w} . We want to emphasize that the path-dependent m th-degree risk premium is a choice-based notion and that the condition for one individual's risk premium to be uniformly larger than another one's is independent of whether preferences can be represented within the expected utility framework.

3.3. The Comparative Statics Approach

Jindapon and Neilson (2007) construct a decision problem, where an individual can reduce the n th-degree risk in the random wealth by incurring a monetary cost. They show that, in the expected utility framework, individual $u(x)$ would always want to incur a larger monetary cost than individual $v(x)$ and hence, further reduce the n th-degree risk in the random wealth if and only if the former is $(n/1)$ th-degree Ross more risk averse than the latter. They refer to their analysis as the comparative statics approach to comparative n th-degree risk aversion.¹³

In the problem of Jindapon and Neilson (2007), the move to further reduce the n th-degree risk in the random wealth by incurring a larger monetary cost can be decomposed into an n th-degree risk decrease (an improvement) and a first-degree risk increase

(a deterioration). The definition below provides a general notion of changes in a random distribution that can be decomposed into an improvement and a deterioration.

Definition 6. $F(x)$ is jointly less n th-degree risky and more m th-degree risky than $G(x)$ if there exists $H(x)$, such that $F(x)$ has less n th-degree risk than $H(x)$ and $H(x)$ has more m th-degree risk than $G(x)$.

Now consider a parameterized wealth path represented by $\tilde{w}(\alpha)$, where as α increases, $\tilde{w}(\alpha)$ becomes jointly less n th-degree risky and more m th-degree risky. Moving down such a path, one reduces the n th-degree risk in wealth by increasing the m th-degree risk in wealth. Intuitively, if one individual always chooses a larger α than another individual for all such parameterized wealth paths, it must be that the former is more n th-degree risk averse than the latter.

Adopting the expected utility framework, the problem of an individual $u(x)$ is

$$\max_{\alpha} Eu[\tilde{w}(\alpha)]. \quad (10)$$

The solution to (10) is assumed to be unique and denoted α_u .¹⁴ Similarly, denote the optimal choice of another individual $v(x)$ as α_v . In the appendix, we prove the following characterization theorem for $\alpha_u \geq \alpha_v$.

Theorem 3. $\alpha_u \geq \alpha_v$ for every wealth path $\tilde{w}(\alpha)$, where as α increases, $\tilde{w}(\alpha)$ becomes jointly less n th-degree risky and more m th-degree risky if and only if $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$.

There are a number of situations that can give rise to wealth paths $\tilde{w}(\alpha)$, where increases in α make wealth jointly less n th-degree risky and more m th-degree risky, and one of such situations is a direct generalization of the problem considered in Jindapon and Neilson (2007).¹⁵ Suppose that total wealth consists of two independent components (i.e., $\tilde{w}(\alpha) = \tilde{w}_1(\alpha) + \tilde{w}_2(\alpha)$), and as α increases, $\tilde{w}_1(\alpha)$ becomes less n th-degree risky and $\tilde{w}_2(\alpha)$ becomes more m th-degree risky. Note that the setup in Jindapon and Neilson (2007) is a special case of this situation, where $m = 1$. It can be immediately checked that, as α increases, $\tilde{w}(\alpha)$ becomes jointly less n th-degree risky and more m th-degree risky. Then, according to Theorem 3, an (n/m) th-degree Ross more risk-averse individual would choose to have a less n th-degree risky first component and a more m th-degree risky second component.

Finally, it is important to point out that the assumption of a unique optimal solution to problem (10) is made for easy exposition rather than technical necessity. Indeed, Theorem 3 as well as its proof in the appendix remain valid if the unique solution to (10) is replaced with a general set-valued solution. Such generalizability is important, because global concavity of the objective function, which is often entailed by the

assumption of a unique optimal solution, may impose strong constraints on the set of admissible utility functions for a given risk change or the set of risk changes for a given utility function.

4. Conclusion

More than half a century ago, Pratt (1964) used two behavioral (or choice-based) conditions—which are based on the risk premium and the probability premium—to characterize the Arrow–Pratt more risk-averse condition that is based on the famous Arrow–Pratt risk aversion measure, $-u''(x)/u'(x)$. These behavioral conditions regarding comparative risk aversion are important both because they have economic contents and because they can be readily implemented in experimental investigations into individual characteristics (e.g., gender, age, income, education, and religion) that affect the degree of risk aversion. These behavioral conditions do not depend on the expected utility framework to be meaningful and can be checked via experiments without explicit specifications of the utility function.

More recently, Liu and Meyer (2013) propose to use $\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(x)}$ as the (n/m) th-degree risk aversion measure for n th-degree risk aversion and generalize the Arrow–Pratt more risk-averse condition and the Ross more risk-averse condition to the (n/m) th-degree Ross more risk aversion.

This paper generalizes the three main existing (behavioral or choice-based) approaches to comparative risk aversion—the probability premium approach and the risk premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007)—for comparative n th-degree risk aversion that can accommodate trading off an n th-degree risk increase and an m th-degree risk increase for any m , such that $1 \leq m < n$. It shows that, when the expected utility framework is assumed, all of these general notions of comparative n th-degree risk aversion are equivalent and can be characterized by the (n/m) th-degree Ross more risk aversion.

In the future, economists and other social scientists as well as financial analysts and decision analysts may want to investigate the determining factors of the strength of third-degree and higher-degree risk aversion, just as they have extensively done for the second-degree risk aversion. It is our hope that the results in this paper will deepen the understanding of and help in creating alternative measures for the intensity of n th-degree risk aversion.

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Appendix. Proofs

Proof of Lemma 3. If $u(x)$ is not (n/m) th-degree Ross more risk averse than $v(x)$ on $[a, b]$, then there exist some y and $z \in [a, b]$, such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \frac{u^{(m)}(z)}{v^{(m)}(z)}.$$

Obviously, for such y and z , there exists $\mu > 0$, such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)},$$

which implies, because of continuity, that there exist $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$, such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)}$$

for all $y \in [a_1, b_1]$ and all $z \in [a_2, b_2]$.

Define $\phi(x) \equiv u(x) - \mu v(x)$. Differentiating yields

$$(-1)^{n+1} \phi^{(n)}(x) = (-1)^{n+1} u^{(n)}(x) - \mu(-1)^{n+1} v^{(n)}(x) < 0$$

for all $x \in [a_1, b_1]$

$$(-1)^{m+1} \phi^{(m)}(x) = (-1)^{m+1} u^{(m)}(x) - \mu(-1)^{m+1} v^{(m)}(x) > 0$$

for all $x \in [a_2, b_2]$. Q.E.D.

Proof of Theorem 1. For any given \tilde{w} , \tilde{y} , and \tilde{z} , such that \tilde{y} is n th-degree more risky than \tilde{w} and \tilde{z} is m th-degree less risky than \tilde{w} , define

$$U(p) \equiv Eu(\tilde{w}) - [pEu(\tilde{z}) + (1-p)Eu(\tilde{y})].$$

Clearly, $U'(p) = Eu(\tilde{y}) - Eu(\tilde{z}) < 0$, because $u(x)$ is both m th-degree risk averse and n th-degree risk averse. For the same reason, $U(0) > 0$ and $U(1) < 0$.

Therefore, $\exists p_u \in (0, 1)$, such that $U(p_u) = 0$. $V(p)$ for $v(x)$ can be similarly defined, and $\exists p_v \in (0, 1)$, such that $V(p_v) = 0$.

For the “if” part, suppose that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$. Then, from Lemma 2, there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1} \phi^{(m)}(x) \leq 0$ and $(-1)^{n+1} \phi^{(n)}(x) \geq 0$ for all x in $[a, b]$, such that $u(x) \equiv \lambda v(x) + \phi(x)$. Note that $\phi(x)$ is both weakly m th-degree risk tolerant—meaning that $-\phi(x)$ is weakly m th-degree risk averse—and weakly n th-degree risk averse.

Evaluating $U(p)$ at p_v , we have

$$\begin{aligned} U(p_v) &= Eu(\tilde{w}) - [p_v Eu(\tilde{z}) + (1-p_v)Eu(\tilde{y})] \\ &= \lambda V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= p_v [E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v) [E\phi(\tilde{w}) - E\phi(\tilde{y})] \\ &\geq 0. \end{aligned}$$

The inequality above holds, because (i) \tilde{z} has less m th-degree risk than \tilde{w} and \tilde{y} has more n th-degree risk than \tilde{w} and because (ii) $\phi(x)$ is both weakly m th-degree risk tolerant and weakly n th-degree risk averse. Because $U(p)$ is strictly decreasing in p , we have $p_u \geq p_v$.

For the “only if” part, suppose that $p_u \geq p_v$ for all \tilde{w} , \tilde{y} , and \tilde{z} , such that \tilde{y} is n th-degree more risky than \tilde{w} and \tilde{z} is m th-degree less risky than \tilde{w} . To prove that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$, assume otherwise. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$, and $[a_2, b_2] \subset (a, b)$, such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} (-1)^{n+1} \phi^{(n)}(x) &< 0 && \text{for all } x \in [a_1, b_1] \\ (-1)^{m+1} \phi^{(m)}(x) &> 0 && \text{for all } x \in [a_2, b_2]. \end{aligned} \tag{A.1}$$

Now denote the CDFs for \tilde{w} , \tilde{y} , and \tilde{z} as $F(x)$, $G(x)$, and $H(x)$, respectively, and choose $F(x)$, $G(x)$, and $H(x)$, such that

$$\begin{cases} G^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ G^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \quad \begin{cases} F^{[m]} - H^{[m]} > 0 & x \in (a_2, b_2) \\ F^{[m]} - H^{[m]} = 0 & x \notin (a_2, b_2). \end{cases} \tag{A.2}$$

Evaluating $U(p)$ at p_v , we have

$$\begin{aligned} U(p_v) &= \mu V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= p_v [E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v) [E\phi(\tilde{w}) - E\phi(\tilde{y})] \\ &= p_v \int_a^b \phi(x) d[F(x) - H(x)] + (1-p_v) \int_a^b \phi(x) d[F(x) - G(x)] \\ &= p_v \int_a^b (-1)^{m+1} \phi^{(m)}(x) [H^{[m]}(x) - F^{[m]}(x)] dx \\ &\quad + (1-p_v) \int_a^b (-1)^{n+1} \phi^{(n)}(x) [G^{[n]}(x) - F^{[n]}(x)] dx \\ &= p_v \int_{a_2}^{b_2} (-1)^{m+1} \phi^{(m)}(x) [H^{[m]}(x) - F^{[m]}(x)] dx \\ &\quad + (1-p_v) \int_{a_1}^{b_1} (-1)^{n+1} \phi^{(n)}(x) [G^{[n]}(x) - F^{[n]}(x)] dx \\ &< 0. \end{aligned}$$

The inequality above is from (A.1) and (A.2). Because $U(p)$ is strictly decreasing in p , we have $p_u < p_v$, a contradiction. Therefore, $u(x)$ must be (n/m) th-degree Ross more risk averse than $v(x)$. Q.E.D.

Proof of Theorem 2. For the “if” part, suppose that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$. Then, according to Lemma 2, there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1} \phi^{(m)}(x) \leq 0$ and $(-1)^{n+1} \phi^{(n)}(x) \geq 0$ for all x in $[a, b]$, such that $u(x) \equiv \lambda v(x) + \phi(x)$. Note that $\phi(x)$ is both weakly m th-degree risk tolerant—meaning that $-\phi(x)$ is weakly m th-degree risk averse—and weakly n th-degree risk averse.

For every \tilde{w} , every \tilde{y} that is n th-degree riskier than \tilde{w} , and every path of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0, B]}$,

π_u and π_v satisfy $Eu(\tilde{x}(\pi_u)) = Eu(\tilde{y})$ and $Ev(\tilde{x}(\pi_v)) = Ev(\tilde{y})$, respectively, by definition. Furthermore, we have

$$\begin{aligned} Eu(\tilde{x}(\pi_v)) &= \lambda Ev(\tilde{x}(\pi_v)) + E\phi(\tilde{x}(\pi_v)) \\ &= \lambda Ev(\tilde{y}) + E\phi(\tilde{x}(\pi_v)) \\ &\geq \lambda Ev(\tilde{y}) + E\phi(\tilde{y}) \\ &= Eu(\tilde{y}), \end{aligned}$$

where the inequality holds, because (i) $\tilde{x}(\pi_v)$ has more m th-degree risk than \tilde{w} and \tilde{y} has more n th-degree risk than \tilde{w} and because (ii) $\phi(x)$ is both weakly m th-degree risk tolerant and weakly n th-degree risk averse. Because $Eu(\tilde{x}(\pi))$ is strictly decreasing in π , we have $\pi_u \geq \pi_v$.

For the “only if” part, suppose that the path-dependent m th-degree risk premia satisfy $\pi_u \geq \pi_v$ for every \tilde{w} , every \tilde{y} that is n th-degree riskier than \tilde{w} , and every path of m th-degree increasing risk from \tilde{w} , $\{\tilde{x}(\pi)\}_{\pi \in [0,1]}$. Then, it must be the case that $\pi_u \geq \pi_v$ for every \tilde{w} , every \tilde{y} that is n th-degree riskier than \tilde{w} , and a special path of m th-degree increasing risk from \tilde{w} that is defined as follows.

Suppose that \tilde{z} (with CDF $H(x)$) has more m th-degree risk than \tilde{w} (with CDF $F(x)$). Then, $\{\tilde{x}(\pi)\}_{\pi \in [0,1]}$ is a path of m th-degree increasing risk from \tilde{w} if $\tilde{x}(\pi)$ has a CDF of $\pi H(x) + (1 - \pi)F(x)$. Additionally, for $u(x)$, the path-dependent m th-degree risk premium for an n th-degree risk increase from \tilde{w} to \tilde{y} along this path is given by $\pi Eu(\tilde{z}) + (1 - \pi)Eu(\tilde{w}) = Eu(\tilde{y})$ or

$$\pi_u = \frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})}.$$

Similarly,

$$\pi_v = \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}.$$

Therefore, the given condition implies that $\frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})} \geq \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}$ for every \tilde{w} , every \tilde{y} that is n th-degree riskier than \tilde{w} , and every \tilde{z} that is m th-degree riskier than \tilde{w} . Note that the ratio on each side of the inequality is the “rate of substitution” between an n th-degree risk increase and an m th-degree risk increase defined in Liu and Meyer (2013). According to theorem 1 of Liu and Meyer (2013), it must be the case that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$. Q.E.D.

Proof of Theorem 3. For the “if” part, suppose that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$. By Lemma 2, there exist $\lambda > 0$ and $\phi(x)$, such that $u(x) \equiv \lambda v(x) + \phi(x)$, where $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all x . Note that $\phi(x)$ is both weakly m th-degree risk tolerant—meaning that $-\phi(x)$ is weakly m th-degree risk averse—and weakly n th-degree risk averse.

We use proof by contradiction. To prove $\alpha_u \geq \alpha_v$, assume $\alpha_u < \alpha_v$ instead. Note that

$$\begin{aligned} Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) &= \lambda [Ev(\tilde{w}(\alpha_u)) - Ev(\tilde{w}(\alpha_v))] \\ &\quad + [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v))]. \end{aligned}$$

The first bracket in the above expression is negative, because the expected utility of $v(x)$ is maximized at α_v . Under the

assumption $\alpha_u < \alpha_v$, the second bracket is nonpositive, because $\phi(x)$ is both weakly m th-degree risk tolerant and weakly n th-degree risk averse and because $\tilde{w}(\alpha_v)$ is sequentially less n th-degree risky and more m th-degree risky than $\tilde{w}(\alpha_u)$. Therefore, $Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) < 0$, which contradicts that α_u is the optimal choice for $u(x)$. Therefore, it must be the case that $\alpha_u \geq \alpha_v$.

For the “only if” part, suppose that $\alpha_u \geq \alpha_v$ for every wealth path $\tilde{w}(\alpha)$, where as α increases, $\tilde{w}(\alpha)$ becomes sequentially less n th-degree risky and more m th-degree risky. To prove that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$, assume otherwise. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$, and $[a_2, b_2] \subset (a, b)$, such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} (-1)^{n+1}\phi^{(n)}(x) &< 0 && \text{for all } x \in [a_1, b_1] \\ (-1)^{m+1}\phi^{(m)}(x) &> 0 && \text{for all } x \in [a_2, b_2]. \end{aligned} \tag{A.3}$$

Because $\alpha_u \geq \alpha_v$, $\tilde{w}(\alpha_u)$ is sequentially less n th-degree risky and more m th-degree risky than $\tilde{w}(\alpha_v)$. Therefore, there exists \tilde{z} , such that $\tilde{w}(\alpha_u)$ has less n th-degree risk than \tilde{z} and \tilde{z} has more m th-degree risk than $\tilde{w}(\alpha_v)$.

Denote the CDFs for $\tilde{w}(\alpha_u)$, $\tilde{w}(\alpha_v)$, and \tilde{z} as $F(x)$, $G(x)$, and $H(x)$, respectively. Because of the arbitrariness of the wealth path $\tilde{w}(\alpha)$, we can choose $F(x)$, $G(x)$, and $H(x)$, such that

$$\begin{cases} H^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ H^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \quad \begin{cases} H^{[m]} - G^{[m]} > 0 & x \in (a_2, b_2) \\ H^{[m]} - G^{[m]} = 0 & x \notin (a_2, b_2). \end{cases} \tag{A.4}$$

Then, we have

$$\begin{aligned} &Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) \\ &= \mu [Ev(\tilde{w}(\alpha_u)) - Ev(\tilde{w}(\alpha_v))] + [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v))] \\ &< E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v)) \\ &= [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{z})] + [E\phi(\tilde{z}) - E\phi(\tilde{w}(\alpha_v))] \\ &= \int_a^b \phi(x)d[F(x) - H(x)] + \int_a^b \phi(x)d[H(x) - G(x)] \\ &= \int_a^b (-1)^{n+1}\phi^{(n)}(x)[H^{[n]}(x) - F^{[n]}(x)]dx \\ &\quad + \int_a^b (-1)^{m+1}\phi^{(m)}(x)[G^{[m]}(x) - H^{[m]}(x)]dx \\ &= \int_{a_1}^{b_1} (-1)^{n+1}\phi^{(n)}(x)[H^{[n]}(x) - F^{[n]}(x)]dx \\ &\quad + \int_{a_2}^{b_2} (-1)^{m+1}\phi^{(m)}(x)[G^{[m]}(x) - H^{[m]}(x)]dx \\ &< 0. \end{aligned} \tag{A.5}$$

The first inequality above is from the fact that α_v is the optimal choice for $v(x)$. The second inequality above is from (A.3) and (A.4). Note that (A.5) implies that α_u is not the optimal choice for $u(x)$, a contradiction. Therefore, it must be the case that $u(x)$ is (n/m) th-degree Ross more risk averse than $v(x)$. Q.E.D.

Endnotes

¹ For examples of choice-based risk aversion measures, see Eckel and Grossman (2002), Holt and Laury (2002), Andreoni and Sprenger

(2011), Callen et al. (2014), Ebert and Wiesen (2014), and Grossman and Eckel (2015).

²For example, see Deck and Schlesinger (2010), Ebert and Wiesen (2011), Maier and Ruger (2011), Deck and Schlesinger (2014), and Noussair et al. (2014).

³For example, see Ross (1981), Machina and Neilson (1987), Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010).

⁴Of course, these three approaches are not exhaustive. For example, Keenan and Snow (2009), Crainich and Eeckhoudt (2011), Liu and Meyer (2012), Li and Liu (2014), Huang and Stapleton (2015), Keenan and Snow (2016), and Keenan and Snow (2017) investigate additional alternative approaches to comparative third-degree or higher-degree risk aversion. In particular, Li and Liu (2014) show that one individual has an everywhere larger ratio of the n th and m th derivatives of the utility function if and only if that individual has a larger monetary utility premium for an n th-degree risk than the other individual, a result very much in the spirit of the ones in this paper. As a behavioral condition, however, the monetary utility premium is not as standard or useful for practitioners as the behavioral conditions that we study here.

⁵The “good” or “bad” is from the perspective of an individual who is both m th-degree risk averse and n th-degree risk averse.

⁶Almost all often-encountered utility functions satisfy this “mixed risk aversion” property (Brockett and Golden 1987, Caballe and Pomansky 1996). Eeckhoudt and Schlesinger (2006) and Menegatti (2015) provide significant new results on how these preferences may be characterized. For example, Menegatti (2015) shows that, if the n th-order derivative of an increasing function $u(x)$ defined on $[0, \infty]$ is sign invariant, then all of the derivatives of orders from 2 to n alternate in sign. Note also that, throughout this paper and for Definition 2 in particular, the outcome variable x is assumed to belong to a bounded interval $[a, b]$. With the assumption of a bounded domain imposed in this paper, the results of Menegatti (2015) do not apply.

⁷See also Denuit et al. (1999) and Jouini et al. (2013).

⁸To appreciate the usefulness of these lemmas in proofs, it may be helpful to discuss their counterparts in the context of the more familiar Arrow–Pratt more risk aversion. By definition, $u(x)$ is Arrow–Pratt more risk averse than $v(x)$ if $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ for all $x \in [a, b]$. Then, the Arrow–Pratt counterpart of Lemma 2 is “ $u(x)$ is Arrow–Pratt more risk averse than $v(x)$ if and only if the transformation function $T(y)$, defined through $u(x) \equiv T(v(x))$, satisfies $T''(y) \leq 0$ for all $y \in [v(a), v(b)]$,” and the Arrow–Pratt counterpart of Lemma 3 is, “if $u(x)$ is not Arrow–Pratt more risk averse than $v(x)$, then there exists $[a_1, b_1] \subset (a, b)$, such that, for the transformation function $T(y)$ defined through $u(x) \equiv T(v(x))$, $T''(y) > 0$ for all $y \in [v(a_1), v(b_1)]$.”

⁹See the proof of theorem 1 of Liu and Meyer (2013).

¹⁰The only exceptions are Jindapon (2010) and Watt (2011), who propose alternative probability-type measures of downside risk aversion based on the risk apportionment framework of Eeckhoudt and Schlesinger (2006). In addition, Eeckhoudt and Laeven (2015) recently gave a graphical representation of the probability premium.

¹¹This original notion of risk premium of Arrow–Pratt and Ross has been used to measure an individual’s aversion to higher-degree risk increases by Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010).

¹²Continuity is for the space of probability distributions and with respect to the topology of weak convergence.

¹³Watt and Vazquez (2013) provide an alternative comparative statics approach to comparative downside risk aversion.

¹⁴Without the expected utility approach, the value of α_i would be defined by the criterion $\tilde{w}(\alpha_i)$, which is weakly preferred to $\tilde{w}(\alpha)$ for all α .

¹⁵Discussions of other situations where Theorem 3 is applicable are available from the authors on request.

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