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# Comparative Risk Sensitivity with Reference-Dependent Preferences

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*Abstract*

Experimental evidence suggests that individuals are risk averse over gains and risk seeking over losses (i.e., they have S-shaped utility functions in an expected utility setting) and that they are loss averse. Furthermore, the evidence leads to a single definition of S-shaped utility, but it has led to several alternative specifications of loss aversion. This paper characterizes the relations "more S-shaped than" and "more loss averse than" for a utility function, and in so doing arrives at a new definition of loss aversion based on average instead of marginal utility.

**Keywords:** risk aversion, loss aversion, reference dependence, prospect theory, expected utility

**JEL Classification:** D81

Experimental work on behavior toward risk, especially that of Kahneman and Tversky (1979) and Tversky and Kahneman (1981, 1991, 1992), establishes that individuals base decisions on gains and losses from some reference level of wealth, a notion known as reference dependence. The alternative hypothesis of asset integration, that is, that individuals base their decisions on final wealth positions, is rejected by the data. Two underlying patterns arise from the data. The first is that individuals are risk averse over gains but risk seeking over losses. This pattern, termed diminishing sensitivity, implies for expected utility maximizers that utility functions are concave over positive changes in wealth and convex over negative changes in wealth, or S-shaped. The second pattern is that losses matter more than corresponding gains. This pattern, termed loss aversion, implies for expected utility maximizers that the utility function is steeper for losses than it is for gains.[1]

Some of the best-known results from expected utility theory, namely those associated with comparative risk aversion, as in Arrow (1974) and Pratt (1964), were built under the guiding assumptions that the utility function is concave and defined over final wealth positions. For example, once one assumes that individuals are risk averse over the entire range of outcomes, it makes sense to discuss one individual being more risk averse than another. Reference dependence change the framework from which questions are asked, though. If individuals are risk averse over gains and risk seeking over losses, it is no longer so compelling to talk about one individual being more risk averse than another. Instead, it makes sense to talk about one individual being more risk averse over gains

and more risk seeking over losses. The presence of loss aversion raises the issue of one individual being more loss averse than another.

The purpose of this paper is to characterize changes in risk attitudes when the assumption of asset integration fails. This is accomplished using a series of equivalence theorems in the spirit of those first advocated by Machina (1983). For each notion of comparative risk attitudes, a utility transformation, a mathematical condition, a risk premium condition, and an asset demand condition are shown to be equivalent (see, e.g., Pratt (1964) and Machina and Neilson (1987)). The utility transformation turns out to be the key to the analysis in this paper. To see why, note that the standard Arrow-Pratt utility transformation condition can be expressed as "$u$ is a risk averse (i.e., concave) transformation of $v$." So, $u$ is more risk averse than $v$ if $u$ is a risk averse transformation of $v$. Following this logic, one can construct an equivalence theorem for the relation "more S-shaped than" based on $u$ being an S-shaped transformation of $v$, and another equivalence theorem for the relation "more loss averse than" based on $u$ being a loss averse transformation of $v$.

The latter is problematic, though, because loss aversion is not yet well defined. Instead, loss aversion is based on the vague notion that losses matter more than gains. When they first introduced it, Kahneman and Tversky (1979) began with the idea that $u$ is loss averse if $u$ is unwilling to accept symmetric fair bets. This implies that $u(x) \leq -u(-x)$ when $x > 0$, and it also implies that $u'_+(0) \leq u'_-(0)$, so that $u$ is steeper for small losses than for small gains.[2] Other interpretations can be found by making the original notions more global. For example, a second interpretation is that $u$ is loss averse if $u$ is unwilling to accept *any* fair bet, and a third interpretation is that $u$ is loss averse if $u'(y) \leq u'(z)$ for all $z < 0 < y$, so that $u$ is *everywhere* steeper for losses than for gains. Bowman, Minehart, and Rabin (1999) use the third interpretation in their analysis of consumption/savings decisions with loss aversion. If one could construct an equivalence theorem for the relation "more loss averse than," it would identify the "correct" definition of loss aversion by comparing $u$ to a risk neutral utility function.

The analysis shows that it is possible to construct an equivalence theorem for "more S-shaped than." The intuition is straightforward. A function is S-shaped if it is concave over gains and convex over losses, and one utility function is more S-shaped than another if the first one is more concave over gains and more convex over losses than the other. An equivalence theorem is constructed relating this notion to a condition regarding the Arrow-Pratt coefficient of risk aversion, a risk premium condition, and an asset demand condition. When all outcomes are gains, the more S-shaped individual has a higher risk premium and a lower demand for the risky asset than the S-shaped individual, consistent with the notion of more risk averse over gains, and when all outcomes are losses, the more S-shaped individual has a lower risk premium and a higher demand for the risky asset, consistent with the notion of less risk averse over losses.

The analysis of loss aversion is more complex, since there is not yet a single definition of loss aversion. Unfortunately, an appropriate risk premium condition is generated by one definition of loss aversion, and an appropriate asset demand condition is generated by a different, stronger definition of loss aversion, leaving the researcher with too many notions of loss aversion. Weak loss aversion, which is consistent with the

risk premium condition, is based on average utility over losses being everywhere greater than average utility over gains. Strong loss aversion, which is consistent with the asset demand condition, is based on marginal utility over losses being everywhere greater than marginal utility over gains. These notions should not come as a surprise. Since risk premia are found from an indifference condition, it makes sense that they would require a restriction on average utilities, while asset demands arise from a marginal condition, and consequently require a restriction on marginal utilities.

This problem of too many definitions of loss aversion is mitigated, however, when the utility function is also S-shaped, because then the two definitions of loss aversion can be shown to be identical. Consequently, there is a single, well-defined notion of "more S-shaped *and* loss averse than," and loss aversion and S-shaped utility are fundamentally linked.

## 1. The approach

Let individuals have preferences over all probability distributions with support contained in a bounded interval $[m, M]$, with $m < 0 < M$. Individuals are assumed to be expected utility maximizers with strictly increasing utility functions of the form $u(x)$, where $x$ denotes a deviation from reference wealth, and $u$ is assumed to be bounded on $[m, M]$. While one might expect to include reference wealth as a second argument of the utility function, it is omitted for notational convenience. The expected utility hypothesis is maintained for several reasons. First, by assuming expected utility, the results presented here can be readily compared to the Arrow-Pratt characterization of comparative absolute risk aversion. Second, this paper is primarily concerned with loss aversion, which so far has been characterized as a property of utility functions, and not a property of probability transformations in nonexpected utility models like rank-dependent expected utility. Accordingly, no probability transformations are used. Finally, assuming expected utility greatly simplifies the analysis.[3]

The Arrow-Pratt characterization of comparative absolute risk aversion provides an appropriate starting point for the analysis both because it establishes a benchmark against which other characterizations can be compared and because it illustrates a method for constructing appropriate notions of comparative risk attitudes. Let $u$ and $v$ be two strictly increasing, twice continuously differentiable utility functions. The notion that $u$ is more risk averse than $v$ can be characterized in four ways:

(A1) $u(x) = f(v(x))$ for some increasing concave function $f$.
(A2) $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ for all $x$.
(A3) For every random variable $\tilde{x}$, letting $\mu$ denote its expected value, let $\pi_u$ and $\pi_v$ solve $u(\mu - \pi_u) = E[u(\tilde{x})]$ and $v(\mu - \pi_v) = E[v(\tilde{x})]$ respectively. Then $\pi_u \geq \pi_v$.
(A4) For any random variable $\tilde{x}$ with $E[\tilde{x}] \geq 0$, let $a_u, a_v \in [0, 1]$ maximize $E[u(y + a_u\tilde{x})]$ and $E[v(y + a_v\tilde{x})]$ respectively. Then $a_u \leq a_v$.

Condition (A1) states that $u$ is more concave than $v$, and condition (A2) says that $u$ has a higher Arrow-Pratt coefficient of risk aversion than $v$ does. These are both mathematical

characterizations. In contrast, (A3) and (A4) are behavioral characterizations. (A3) states that $u$ is willing to pay a higher risk premium to avoid a risk than $v$ is, and (A4) states that upon encountering an opportunity to invest in a riskless asset and a risky one with higher expected return, $u$ invests less in the risky asset than $v$ does. Arrow (1974) and Pratt (1964) show that conditions (A1)–(A3) are equivalent, and that if $u$ and $v$ are both risk averse, these three conditions are also equivalent to (A4). Thus, any of the four conditions can be used to define the relation "more risk averse than."

Importantly, conditions (A1)–(A4) can also be used to define the concept "risk averse" by letting $v$ be a risk neutral utility function. Condition (A1) then implies that $u$ is risk averse if it is concave. This suggests an alternative approach for characterizing the relation "more risk averse than." First, as is standard, define risk aversion by saying that the function $f$ is risk averse if it is increasing and concave, and then replace (A1) by $u(x) = f(v(x))$ for some *risk averse* function $f$. The conditions (A2)–(A4) are then characterizations of the relation "more risk averse than" that are equivalent to $u$ being a risk averse transformation of $v$. This is the approach used here to develop notions of "more S-shaped than" and "more loss averse than." In other words, the starting point is to define what it means for a function to be S-shaped or loss averse, and then find behavioral conditions that are equivalent to $u$ being an S-shaped or loss averse transformation of $v$.

## 2. S-shaped utility

One pattern that has emerged from experimental evidence is that individuals are risk averse over gains and risk seeking over losses. This pattern has been termed *diminishing sensitivity*, because each incremental increase in wealth has less impact than the previous incremental increase, and each incremental decrease in wealth has less impact than the previous incremental decrease. This pattern has obvious implications for the shape of the utility function $u$: it is concave when $x > 0$ and convex when $x < 0$. Graphically, the function is S-shaped, which leads to the following definition.

**Definition.** A function $f(x)$ is *S-shaped* if it is strictly increasing with $f(0) = 0$ and $f''(x) \leq 0$ for all $x > 0$ and $f''(x) \geq 0$ for all $x < 0$.

Here, and in the remainder of the paper, functions are assumed to be differentiable to an appropriate degree everywhere except possibly at zero, and continuous at zero.

Making an S-shaped utility function more concave makes it more risk averse, as shown by the equivalence of conditions (A1)–(A4), but it does not make it more S-shaped. To become more S-shaped, a utility function must become more concave over gains but less concave over losses. A characterization of "more S-shaped than" is provided by the following theorem.

**Theorem 1.** *Let $u$ and $v$ be two increasing utility functions with $u(0) = v(0) = 0$ that are continuous everywhere and twice continuously differentiable except when $x = 0$.*

*Then the following three conditions are equivalent.*

(B1) $u(x) = f(v(x))$ *for some S-shaped function* $f$.
(B2) $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ *for all* $x > 0$ *and* $-u''(x)/u'(x) \leq -v''(x)/v'(x)$ *for all* $x < 0$.
(B3) *Let* $\tilde{x}$ *be a random variable with expected value* $\mu$, *and let* $\pi_u$ *and* $\pi_v$ *solve* $u(\mu - \pi_u) = E[u(\tilde{x})]$ *and* $v(\mu - \pi_v) = E[v(\tilde{x})]$ *respectively. If* $Prob\{\tilde{x} > 0\} = 1$ *then* $\pi_u \geq \pi_v$, *and if* $Prob\{\tilde{x} < 0\} = 1$ *then* $\pi_u \leq \pi_v$.

*In addition, if u and v are S-shaped, (B1)–(B3) are equivalent to the following:*

(B4) *Let* $\tilde{x}$ *be a random variable, and let* $a_u$ *and* $a_v$ *maximize* $E[u(y + a_u\tilde{x})]$ *and* $E[v(y + a_v\tilde{x})]$ *respectively. If* $Prob\{y + \tilde{x} > 0\} = 1$ *and if* $y > 0$, *then* $a_u \leq a_v$. *If* $Prob\{y + \tilde{x} < 0\} = 1$ *and if* $y < 0$, *then* $a_u \geq a_v$.

Theorem 1 is a fairly straightforward extension of the Arrow-Pratt characterization of comparative absolute risk aversion, and its proof, which is excluded, is a fairly straightforward extension of their proofs. Condition (B1) states that $u$ can be obtained from $v$ through an S-shaped transformation, and is therefore more S-shaped that $v$. This condition is also discussed in Wakker and Tversky (1993). (B2) says that $u$ is more risk averse than $v$ (in the Arrow-Pratt sense) when outcomes are gains and more risk seeking when outcomes are losses. (B3) is a risk premium condition, stating that $u$ is willing to pay a higher risk premium to avoid a gamble when the outcomes are gains and is willing to pay a higher surcharge to play the gamble (a more negative risk premium to avoid the gamble) when outcomes are losses.

Condition (B4) is an asset demand condition, and while it coincides with (A4) for gains, it differs in important ways for losses. First, if the utility functions are S-shaped, they are risk seeking over losses, and therefore an asset with higher risk and higher expected return would be strictly preferred to the riskless asset. Consequently, the risky asset in (B4) has lower expected return so that a risk seeking investor would be willing to hold both assets. Second, risk seeking ensures the appropriate second order condition for maximization with these assets. Intuitively, the loss case is simply an arithmetic inverse of the gains case using the function $u^*(-z) = -u(z)$ for $z < 0$. The condition says that if the individual can invest in a riskless asset with a guaranteed negative payoff and a risky asset with lower expected payoff, the more S-shaped individual (or, in this case, the more risk seeking individual) invests more in the risky asset.

One important implication of Theorem 1 is that S-shaped utility does not, by itself, have any ramifications for lotteries that involve both positive and negative outcomes. Behavior in this region is covered by loss aversion, which is the topic of the next section.

## 3. Loss aversion

The definition of loss aversion is less clear-cut than the definition of S-shaped. As stated in the introduction, the literature contains several notions of loss aversion, and it is

possible to construct more. In keeping with the Arrow-Pratt benchmark of conditions (A1)–(A4), the desire is to find a definition of loss aversion that leads a more loss averse utility function to pay a higher risk premium in appropriate situations and to invest less in the risky asset in appropriate situations. Two definitions are offered, beginning with the weaker one.

**Definition.** An increasing function $f(x)$ exhibits *weak loss aversion* if $f(0) = 0$ and $f(y)/y \leq f(z)/z$ for all $z < 0 < y$.

This definition provides one way of capturing the notion that losses matter more than corresponding gains. First, it implies that $f(x) \leq -f(-x)$ for all $x > 0$, which is an even weaker notion of loss aversion and coincides with Kahneman and Tversky's (1979) original notion of loss aversion. Second, a graphical interpretation of weak loss aversion is that a chord connecting the origin to $f(z)$ for any $z < 0$ is steeper than a chord connecting the origin to $f(y)$ for any $y > 0$. Third, and finally, the definition can be thought of in terms of *average* utility rather than the standard marginal utility. The utility function is loss averse if the maximal average utility from any gain is less than the minimal average disutility from any loss.[4]

In the same manner that a definition of "S-shaped" led to a notion of "more S-shaped than," the definition of weak loss aversion leads to a notion of "more weakly loss averse than."

**Theorem 2.** *Let $u$ and $v$ be two increasing functions with $u(0) = v(0) = 0$. The following three conditions are equivalent.*

(C1) $u(x) = f(v(x))$ *for some weakly loss averse function* $f$.
(C2) $-u(y)/u(z) \leq -v(y)/v(z)$ *for all* $z < 0 < y$.
(C3) *For a random variable* $\tilde{x}$, *let* $c_u$ *and* $c_v$ *solve* $u(c_u) = E[u(\tilde{x})]$ *and* $v(c_v) = E[v(\tilde{x})]$. *Then* $c_v = 0$ *implies that* $c_u \leq 0$.

Condition (C1) states that the utility function $u$ is more weakly loss averse than $v$. Condition (C2) says that fixing any gain $y$ and loss $z$, the ratio of the utility of the gain to the disutility of the loss is smaller for $u$ than it is for $v$. (C3) is the only behavioral condition in the set, and it is a certainty equivalent condition. Suppose that there is a random variable $\tilde{x}$ such that $v$'s certainty equivalent of $\tilde{x}$ is zero. Condition (C3) states that for such a random variable the utility function $u$ assigns a lower certainty equivalent. This condition can be transformed into a risk premium condition in the usual way, and $u$ assigns a higher risk premium than $v$ does.

There is no asset demand condition, because the appropriate asset demand condition requires a stronger notion of comparative loss aversion, one that involves derivatives. Accordingly, define strong loss aversion as follows.

**Definition.** An increasing function $f(x)$ exhibits *strong loss aversion* if $f(0) = 0$ and $f'(y) \leq f'(z)$ for all $z < 0 < y$.

Clearly, this definition captures a notion that the function is steeper for losses than for gains. In this case, the function is steeper at every loss than it is at any gain, which means that the function must have a kink at the origin.

Just as weak loss aversion says that the average utility of a gain is less than the average disutility from a loss, strong loss aversion states that the marginal utility of a gain is less than the marginal disutility from a loss. The next proposition establishes that strong loss aversion is, in fact, stronger than weak loss aversion.

**Proposition 1.** *Strong loss aversion implies weak loss aversion.*

Figure 1 establishes that a function can satisfy weak loss aversion without satisfying strong loss aversion, so that the two notions are not equivalent. The function is piecewise linear, and strictly linear over the loss domain. The dashed line is $f'(z)y$ for $z < 0$ and $y > 0$, and since $f(y) < f'(z)y$ for all $y > 0$, the function is weakly loss averse. However, $f'(y_0) > f'(z)$, so the function is not strongly loss averse.

Strong loss aversion leads to a notion of comparative loss aversion that is strong enough for an asset demand condition, but too strong for a risk premium condition, as shown in the next theorem. The theorem requires a definition of diversification, which is the second order condition for the asset demand problem:[5]

**Definition.** An expected utility maximizer with utility function $u$ is a *diversifier* if when $E[\tilde{x}] \geq 0$, $E[\tilde{x}u'(\alpha\tilde{x})]$ is strictly decreasing in $\alpha$.

Suppose that the investor can allocate wealth between two different assets. One asset is riskless and generates a sure (net) return of zero. The other asset is risky and has random return given by $\tilde{x}$. If he invests the amount $\alpha$ of his wealth in the risky asset, the net return on the portfolio is $\alpha\tilde{x}$. The condition in the definition of diversifier guarantees that $E[u(\alpha\tilde{x})]$ is strictly concave in $\alpha$, providing the second order sufficient condition for a maximum.

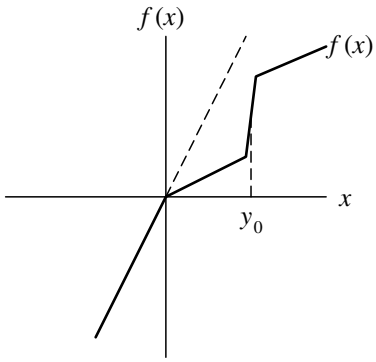


*Figure 1.* A function that exhibits weak but not strong loss aversion.

**Theorem 3.** *Let $u$ and $v$ be two increasing differentiable functions with $u(0) = v(0) = 0$. The following two conditions are equivalent.*

(C1′) $u(x) = f(v(x))$ *for some strongly loss averse function* $f$.
(C2′) $u'(y)/u'(z) \leq v'(y)/v'(z)$ *for all* $z < 0 < y$.

*In addition, if $u$ and $v$ are diversifiers, the above conditions are equivalent to:*

(C4) *Let $\tilde{x}$ be a random variable for which Prob$\{\tilde{x} < 0\} > 0$ and $E[\tilde{x}] \geq 0$. Let $a_u$ and $a_v$ maximize $E[u(a_u\tilde{x})]$ and $E[v(a_v\tilde{x})]$ respectively. Then $a_u \leq a_v$.*

Condition (C1′) states that the utility function $u$ is more strongly loss averse than $v$. Condition (C2′) says that fixing any gain $y$ and loss $z$, the ratio of the marginal utility of the gain to the marginal utility of the loss is smaller for $u$ than it is for $v$. Condition (C4) is the asset demand condition corresponding to the notion of "more loss averse than." An investor chooses to allocate wealth between a riskless asset with net return zero and a risky asset with a nonegative expected return but a positive probability of a negative return. The conditions on the risky asset guarantee that the two assets are not comparable using first-order stochastic dominance, but the condition on the riskless asset is restrictive. (C4) states that the more strongly loss averse individual invests less in the risky asset than the other individual does.

By Proposition 1, conditions (C1′), (C2′) and (C4) each imply the risk premium condition (C3). However, none of them are implied by (C3). Also, note that (C1′) is implied by (A1), since a risk averse function is also loss averse.

## 4. Loss aversion and S-shaped utility together

Considerations of loss aversion and S-shaped utility separate random variables into three distinct classes: those with only gains, those with only losses, and those with both. Because of this, consideration of only one of the properties in the absence of the other incompletely characterizes behavior, and loss aversion and S-shaped utility should be considered as a joint hypothesis.

In the preceding section, characterizations of the relation "more loss averse than" were incomplete in the sense that the risk premium condition and the asset demand condition were not equivalent, with the asset demand condition being stronger than the risk premium condition. Also, two notions of loss aversion were defined—a strong notion based on marginal utilities and a weak notion based on average utilities. Fortunately, the two definitions are equivalent when utility is S-shaped, making it unnecessary to choose between them.

**Theorem 4.** *An S-shaped function defined on $\mathbb{R}$ is weakly loss averse if and only if it is strictly loss averse.*

Theorem 4 allows one to say that an S-shaped function is *loss averse* if it is weakly loss averse or, equivalently, strictly loss averse. From here it is straightforward to apply Theorems 1–4 to generate the following result.

**Corollary.** *Let $u$ and $v$ be two increasing functions that are everywhere continuous and twice continuously differentiable except possibly at 0, and assume that $u(0) = v(0) = 0$. Then the following three conditions are equivalent:*

(D1) $u(x) = f(v(x))$ *for some S-shaped, loss averse function* $f$.
(D2) *Conditions* (B2) *and* (C2).
(D3) *Conditions* (B3) *and* (C3).

*If, in addition, $u$ and $v$ are S-shaped and loss averse, then the above conditions are equivalent to:*

(D4) *Conditions* (B4) *and* (C4).

## 5. Conclusion

This paper constructed counterparts of the relation "more risk averse than" that apply when individuals evaluate gains and losses from reference wealth rather than final wealth positions. Experimental evidence suggests that preferences are risk averse over gains and risk seeking over losses (S-shaped utility) and averse to fair gambles (loss aversion). Accordingly, the relations "more S-shaped than" and "more loss averse than" were characterized. A single characterization of more S-shaped than was developed, but two different notions of more loss averse than arose from the analysis of the behavior conditions. It is possible, though, to provide a single characterization of more S-shaped and loss averse than.

These notions are useful for comparative statics analysis in two ways. The first is obvious—the characterization of more S-shaped and loss averse than allows the researcher to determine the effect when the individual's risk attitudes change. The second arises from a straightforward extension of the analysis to the notion that utility becomes less S-shaped and less loss averse as reference wealth rises. This would be the counterpart of declining risk aversion for utility functions that are reference-dependent.

## Appendix

*Proof of Theorem 2*

(C1) $\Rightarrow$ (C3). Let $\tilde{x}$ be a random variable with $E[v(\tilde{x})] = 0$. By definition, if $f$ is weakly loss averse then there exists a positive real number $c$ such that $f(y)/y \leq c \leq f(z)/z$ for all $z < 0 < y$. This implies that $f(x) \leq cx$ for all $x$. Then

$$E[u(\tilde{x})] = E[f(v(\tilde{x}))] \leq E[cv(\tilde{x})] = 0.$$

(C3) $\Rightarrow$ (C2). Suppose not. Then there exists $z < 0 < y$ and $q \in (0, 1)$ such that $-u(y)/u(z) > (1-q)/q = -v(y)/v(z)$. Let $\tilde{x}$ be a random variable which assigns probability $q$ to outcome $y$ and probability $1-q$ to outcome $z$. Then cross multiplication yields $qu(y) + (1-q)u(z) > 0 = qv(y) + (1-q)v(z)$ which contradicts (C3).

(C2) $\Rightarrow$ (C1). Suppose not, that is, suppose that there exists $z < 0 < y$ such that $f(y)/y > f(z)/z$. Define $y'$ by $v(y') = y$ and define $z'$ by $v(z') = z$. Then

$$-\frac{u(y')}{u(z')} = -\frac{f(v(y'))}{f(v(z'))} > -\frac{v(y')}{v(z')},$$

which contradicts (C2).

*Proof of Proposition 1*

Let $s = \sup\{f'(y) | y > 0\}$. Then $f(y)/y \leq s$ for all $y > 0$. Also, $f'(z) \geq s$ for all $z < 0$. Integrating from 0 to $z$ yields

$$f(z) = \int_0^z f'(x)\,dx \geq \int_0^z s dx = sz.$$

Thus, $f(z)/z \geq s \geq f(y)/y$ for all $z < 0 < y$.

*Proof of Theorem 3*

(C1′) $\Rightarrow$ (C4). Since $u$ is a diversifier, by the definition of $\alpha_u$,

$$0 = \frac{d}{d\alpha} E[u(\alpha\tilde{x})]|_{\alpha_u} = E[\tilde{x}u'(\alpha_u\tilde{x})] = E[\tilde{x}f'(v(\alpha_u\tilde{x}))v'(\alpha_u\tilde{x})].$$

By the definition of strong loss aversion, there exists a constant $c > 0$ such that $f'(x) \geq c$ when $x < 0$ and $f'(x) \leq c$ when $x > 0$. Then $wf'(x) \leq wc$ for any $w, x$ such that $wx > 0$. Therefore

$$0 = E[\tilde{x}f'(v(\alpha_u\tilde{x}))v'(\alpha_u\tilde{x})] \leq cE[\tilde{x}v'(\alpha_u\tilde{x})] = c\frac{d}{d\alpha}E[v(a\tilde{x})]|_{\alpha_u}.$$

Since

$$\frac{d}{d\alpha}E[v(\alpha\tilde{x})]|_{\alpha_u} \geq 0,$$

and since $E[\tilde{x}v'(\alpha\tilde{x})]$ is decreasing in $\alpha$ by assumption, we have $\alpha_u \leq \alpha_v$.

(C4) $\Rightarrow$ (C2′). Suppose not. That is, suppose there is some $z_0 < 0 < y_0$ such that $u'(y_0)/u'(z_0) > v'(y_0)/v'(z_0)$. Choose $y$, $z$, and $p$ so that $a_v y = y_0$, $a_v z = z_0$, and

$$pv'(a_v y)y + (1-p)v'(a_v z)z = pv'(y_0)y + (1-p)v'(z_0)z = 0.$$

Rearranging the above expression yields

$$-\frac{(1-p)z}{py} = \frac{v'(y_0)}{v'(z_0)} < \frac{u'(y_0)}{u'(z_0)},$$

which can be further rearranged to yield

$$pu'(a_v y)y + (1-p)u'(a_v z)z > 0.$$

The second order condition for maximization, which is guaranteed by loss aversion, then implies that $a_u > a_v$, which is a contradiction.

(C2′) ⇒ (C1′). Suppose not. That is, suppose that there is some $z_0 < 0 < y_0$ for which $f'(y_0) > f'(z_0)$. Choose $y_1$ and $z_1$ so that $v(y_1) = y_0$ and $v(z_1) = z_0$. Then $f'(v(y_1)) > f'(v(z_1))$, implying that

$$\frac{u'(y)}{u'(z)} = \frac{f'(v(y))}{f'(v(z))} \cdot \frac{v'(y)}{v'(z)} > \frac{v'(y)}{v'(z)},$$

which is a contradiction.

*Proof of Theorem 4*

Proposition 1 establishes the "if" part of the statement. Now suppose that $f(y)/y \leq f(z)/z$ for all $z < 0 < y$ but that, contrary to the claim, $f'(y_0) > f'(z_0)$ for some $z_0 < 0 < y_0$. Since $f$ is S-shaped, it is convex over losses and $f(z) \leq f(z_0) + f'(z_0)(z - z_0)$ for all $z \leq z_0$. Also, since $f$ is concave over gains, $f'_+(0) \geq f'(y)$ for all $y > 0$. Since $f'_+(0) > f'(z_0)$, there exists some $z^* < 0$ such that $f(z) < f'_+(0)z$ for all $z < z^*$. Since $f'_+(0) = \lim_{x \searrow 0} f(x)/x$, there exists $y^* > 0$ such that for $z < z^*$ and $0 < y < y^*$, $f(z)/z < f(y)/y$, which provides a contradiction.

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## Notes

1. Both diminishing sensitivity and loss aversion can hold outside of the realm of expected utility. In the original version of prospect theory, Tversky and Kahneman capture the two effects using the utility function, using the weighting function to capture other effects. Strictly speaking, though, diminishing sensitivity and loss aversion necessarily hold implications for the shape of the utility function only when the decision-maker is an expected utility maximizer, because then all risk attitudes are captured by the shape of the utility function.

2. $u'_+(0)$ and $u'_-(0)$ denote the right-hand and left-hand derivatives of $u$ at zero, respectively.
3. For examples of analyses of risk attitudes separating utility function effects from probability transformation effects, see Hilton (1988) and Wakker (1994).
4. Weak loss aversion is related to Landsberger and Meilijson's (1990a, b) notion of a star-shaped utility function. A utility function is *star-shaped* at a point $x_0$ if $[u(x) - u(x_0)]/(x - x_0)$ is a nonincreasing function of $x$ on $[m, 0)$ and on $(0, M]$. So, if a function is star-shaped at zero, average utility $u(x)/x$ is nonincreasing on the two intervals. Weak loss aversion is a different requirement, that average utility anywhere in $[m, 0)$ is greater than average utility everywhere in $(0, M]$.
5. For a thorough analysis of diversifiers in a general setting, see Dekel (1989).

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