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Smooth indifference sets*

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Smoothness of indifference sets is proposed as a weaker notion of smoothness for preference orderings. The usual notion of C^r preference orderings is consistent with C^r utility representations and C^{r-1} Marshallian demand functions. Although preferences with indifference sets which are C^r manifolds do not possess C^r utility representations, in general, many of the usual tools from the analysis of smooth utility functions can still be used, such as marginal rates of substitution. The assumption of smooth indifference sets is consistent with smooth Hicksian demand functions but not smooth Marshallian demand functions.

1. Introduction

Ever since the ground-breaking paper on regular economies by Debreu (1970), the tools of differential topology have been used to study general equilibrium theory. Differential methods have led not only to new existence theorems, but also to studies of uniqueness and stability of equilibria.[1] One of the main assumptions of this approach has been that demand functions are smooth. Since demand is a derived function, efforts have been made to find conditions on utility functions and preference orderings which guarantee that demand functions will be smooth. Debreu (1972) solved this problem by developing smoothness conditions for preference orderings.

A preference ordering is said to be smooth if it can be represented by a smooth utility function. Debreu proposed as an alternative definition that preferences are smooth if the indifference graph $\{(x,y)\in X\times X \mid x\sim y\}$, that is, the set of all pairs of indifferent points in consumption space X, is a smooth manifold in $X\times X$. He then demonstrated that the two notions of smooth preferences are equivalent by showing that the preference ordering can be represented by a smooth utility function without critical points if and only if it is monotone, continuous and smooth.

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[1]For a concise account of the uses of differential topology in economic theory, see Debreu (1976b). For a more complete exposition, see Mas-Colell (1985).

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Debreu went on to show that if preferences are smooth and other monotonicity and convexity conditions are satisfied, then (Marshallian) demand functions will be smooth in prices and income. In this sense smooth preferences are consistent with smooth demand functions. It is the purpose of this paper to examine conditions on preferences which are consistent with smooth Hicksian (income-compensated) demand functions. It will be demonstrated that a weaker assumption on preferences can be used (along with the same monotonicity and convexity assumptions) to generate smooth Hicksian demand functions.

The assumption of smooth preferences is stronger than the assumption that indifference sets are smooth manifolds. Instead, it requires that indifference sets be smooth and 'vary smoothly'. Indeed, Dekel (1986) showed that it is possible to have a preference ordering where individual indifference sets are hyperplanes, and therefore trivially smooth, without the preference ordering being representable by a smooth utility function. While the assumption of smooth indifference sets is not strong enough to generate smooth Marshallian demand functions or even smooth utility representations, it is strong enough to generate smooth Hicksian demand functions.

A significant intermediate result from this analysis is that the notion of marginal rates of substitution being given by the slopes of the indifference sets is correct. If indifference sets are smooth manifolds, then they possess tangent hyperplanes at each point. These tangent hyperplanes can be used to construct linear approximations, which in turn have well-defined marginal rates of substitution. These marginal rates of substitution can then be used in the usual fashion. The construction is based on the existence of a slope of the indifference set, which does not depend upon the existence of a smooth utility representation. Instead it depends only on the characteristics of the indifference set in question. The existence of marginal rates of substitution can then be used to derive first-order conditions for maximization of utility subject to a constraint. Using these first-order conditions, Marshallian and Hicksian demand functions can be constructed. It will be shown that the Hicksian demand functions will be smooth in prices, which will not be true in general for the Marshallian demand functions.

If preferences are continuous, it is possible to approximate them arbitrarily closely by a smooth preference ordering [see Mas-Colell (1974)]. The assumption discussed in this paper is weaker than smoothness but stronger than continuity, so the non-smooth preferences could be approximated by smooth ones. There are two reasons to study the weaker class of preferences in spite of this. The first is that the weaker assumption on preferences is sufficient to generate many of the traditional results from consumer theory. This serves to highlight the strength of 'indifference curve analysis' as opposed to 'utility function analysis' in the sense that the properties of individual indifference sets are, in many cases, sufficient to determine behavior without analyzing the entire utility function. The second reason is that the stronger notion of smoothness is not the weakest notion of smoothness that is consistent with smooth Hicksian demand functions.

The paper proceeds as follows: Section 2 introduces the mathematical concepts to be used throughout the work. In section 3 it is demonstrated that if all indifference sets are smooth manifolds, then for each indifference set a surrogate utility function can be constructed so that the surrogates are smooth and share both the indifference set in question and the direction of increasing preference. Since the surrogate is smooth, it can be used to calculate derivatives at appropriate points. In particular, marginal rates of substitution for the non-smooth utility function can be derived from appropriately defined surrogates. These surrogates are used in section 4 to discuss first-order conditions in constrained maximization problems. Section 5 explores smoothness properties of Hicksian demand functions. Finally, the paper concludes in section 6 where it is shown that the actual non-smooth utility function must take a specific functional form similar to that in Dekel (1986).

2. Mathematical notions of smoothness

This section begins with a list of definitions of the relevant mathematical and economic concepts. These are then used to state Debreu's results explicitly and to define the new assumption on preferences.

Let consumption space X be a non-empty open subset of R^n endowed with the usual topology, where n can be considered as the number of commodities. For simplicity we will let $X=R^n_{++}$, the strictly positive orthant of R^n. An open set is needed because differentiability is a local property, that is, it is a property which is defined using open neighborhoods of points in the relevant space.

A function f mapping an open subset of X into R^m is differentiable at x if there is a linear function $Df(\cdot;x)$ mapping R^n into R^m such that $\|f(y)-f(x)-Df(y-x;\, x)\|=o(\|y-x\|)$, where $o(\cdot)$ is a function which is zero at zero and of higher order than its argument. The linear function $Df(\cdot;x)$ is called the derivative of f at x. The derivative provides a linear approximation to f at x. If the derivative at x exists, it is unique. The function is differentiable if it is differentiable at every point in the domain.

If the function $Df(\cdot;x)$ is continuous in x, then f is said to be continuously differentiable, or C^1. [As an alternative definition, consider the vector $\mathrm{d}f(x)$ defined so that $Df(z;x)=\mathrm{d}f(x)\cdot z$. f is C^1 if the function $\mathrm{d}f$ is continuous.] If $Df(\cdot;x)$ is continuously differentiable in x, then f is said to be twice continuously differentiable, or C^2. In general, a function is C^r if it is r times continuously differentiable. The term *smooth* means that the function is at least C^1.

A function f is a C^r diffeomorphism if it is one-to-one, onto, and both f and f^{-1} are C^r. Two sets A and B are said to be C^r diffeomorphic if there exists a C^r diffeomorphism mapping one set onto the other.

A set $M \subset R^n$ is a k-dimensional C^r manifold if it is locally diffeomorphic to R^k, that is, for every $x \in M$ there is an open neighborhood U_x of x, a C^r diffeomorphism g_x mapping R^n into R^n, and a k-dimensional hyperplane H in R^n such that g_x maps $U_x \cap M$ onto an open subset of H. Intuitively, a k-dimensional C^r manifold is a set which, up to a diffeomorphism, has all the properties of a k-dimensional hyperplane. In other words, a smooth manifold is a set which (locally) can be transformed diffeomorphically into a lower dimensional hyperplane and back.

Smooth manifolds have tangents at each point. Let M be a k-dimensional C^r manifold in R^n, as above. The tangent space at x is defined as the image of the map $Dg_x^{-1}(\cdot; g_x(x))$. Recall that $Dg_x^{-1}(\omega; g_x(x))$ is the derivative of the function g_x^{-1} at $g_x(x)$ evaluated at ω. The unique tangent plane to M at x is then the set $x + Dg_x^{-1}(H; g_x(x))$.

A preference ordering $\geqq_p$ over the consumption set X is a binary relation which is reflexive ($x \geqq_p x$ for all $x \in X$), complete (for all $x, y \in X$, either $x \geqq_p y$ or $y \geqq_p x$), and transitive (if $x \geqq_p y$ and $y \geqq_p z$, then $x \geqq_p z$). The weak preference relation $x \geqq_p y$ is interpreted as 'x is at least as good as y'. The associated strict preference relation is denoted by the symbol '$>_p$' and indifference is denoted by '$\sim$'.

An alternative but equivalent characterization of the preference ordering $\geqq_p$ is as a subset P of $X \times X$. Then $(x, y) \in P$ if and only if $x \geqq_p y$.

The preference ordering $\geqq_p$ is continuous if the sets $\{x | x \geqq_p y\}$ and $\{x | y \geqq_p x\}$ are closed for all $y \in X$. A well-known result is that continuous preference orderings can be represented by continuous utility functions, that is, if $\geqq_p$ is continuous then there exists a continuous real-valued function u defined on X such that $u(x) \geqq u(y)$ if and only if $x \geqq_p y$ [see, for example, Debreu (1954) or Barten and Bohm (1982)].

The preference ordering $\geqq_p$ is said to be monotone if $x \geqq y$ and $x \neq y$ implies $x >_p y$. If preferences are monotone and if bundle x has at least as much of every commodity and strictly more of some commodity than bundle y, then x must be strictly preferred to y. This is a stronger assumption on preferences than, say, local non-satiation, which says that for any point in consumption space there is another point arbitrarily close to it which is strictly preferred to it. However, monotonicity is more useful in the subsequent arguments [see also Mas-Colell (1985)].

The preference ordering $\geqq_p$ is said to be convex if the set $\{x | x \geqq_p y\}$ is convex for all $y \in X$. It is strictly convex if $\{x | x \geqq_p y\}$ is strictly convex for all $y \in X$ [see, for example, Intriligator (1971)]. The associated concepts for utility functions are quasiconcavity and strict quasiconcavity, respectively.

Debreu (1972) examined conditions on preferences which would allow them to be represented by C^r utility functions. He showed that $\geqq_p$ can be represented by a C^r utility function without critical points if and only if $\geqq_p$ is monotone and continuous and the set $\partial P \equiv \{(x,y) \in X \times X \,|\, x \sim y\}$ is a C^r manifold in R^{2n}. Because of this equivalence, we can say that the preference relation is C^r if the set ∂P is C^r manifold.

We now look at a weaker notion of smoothness for preference orderings. The above notion requires not only that individual indifference sets be smooth, but that the entire *indifference graph* be smooth. In this sense indifference sets must be smooth and 'vary smoothly'. The weaker notion of smoothness comes from simply dropping the requirement that the indifference sets vary smoothly.

Definition. A preference ordering over R^n_{++} is *weakly* C^r if all its indifference sets are $n-1$ dimensional C^r manifolds in R^n_{++}.

Of course, weakly smooth preferences could be defined on other n-dimensional open C^r manifolds in R^n.

If $\geqq_p$ is a monotone, continuous, weakly smooth preference ordering, then it may not have a smooth utility representation. However, it will have a well-defined, unique marginal rate of substitution at each point. This is not surprising because in undergraduate microeconomics texts, the marginal rate of substitution is defined as the (negative of the) slope of the indifference curve. The assumption that preferences are weakly smooth guarantees that the indifference curves have slopes, and furthermore, the slopes are unique.

Formally, if the indifference set I is an $n-1$ dimensional C^r manifold, then for each $x \in I$, there exists a unique $n-1$ dimensional tangent space T_x through x. There exists a continuous linear functional L_x mapping R^n into the real line and some real number a such that $L_x(y)=a$ if and only if $y \in T_x$. It can be assumed that L_x is increasing in each argument. Associated with the linear functional L_x is an n-dimensional vector μ_x such that $L_x(y)=\mu_x \cdot y$. This vector μ_x corresponds to the vector of marginal utilities for a smooth utility function. As with a vector of marginal utilities, the elements of μ_x can be used to find marginal rates of substitution between various commodities [see, for example, Barten and Bohm (1982)].

Examples of smooth preference orderings abound. Linear preferences are smooth, as are those represented by Cobb–Douglas utility functions. By Proposition 1 below, any homogeneous degree one function with smooth indifference sets is also smooth. Leontief preferences are neither smooth nor weakly smooth. The reason is that individual indifference sets, which are L-shaped, are not smooth manifolds at the corners. An example of preferences which are weakly smooth but not smooth can be found in Dekel (1986). Let there be two commodities, x_1 and x_2. By Billingsley (1986, Example 31.1) there exists a function $f\colon [0,\frac{1}{2}] \to [0,a]$ for some $a \in (\frac{1}{2},1]$ which is continuous and strictly increasing with derivative zero almost everywhere. Indifference sets are defined by drawing straight lines between the point z on the x_1 axis and $f(z)$ on the x_2 axis. Preferences are weakly smooth because indifference curves are line segments, but Dekel showed that the preference ordering cannot be smooth. Let u be a utility function representing these preferences. Then $u(z,0)\equiv u(0,f(z))$ for $z\in[0,\frac{1}{2}]$. If u is differentiable, then $u_1(z,0)=u_2(0,f(z))\cdot f'(z)$ wherever f is differentiable. This implies that $u_1(z,0)=0$ almost everywhere, so that $u(z,0)$ is constant for $z\in[0,\frac{1}{2}]$. This contradicts the assumption that the preference ordering is strictly increasing along the x_1 axis, and so u cannot be differentiable.[2]

3. Using smooth indifference manifolds

This section begins by giving an example of how, given a single indifference set, it is possible to construct an indifference map where the indifference curves 'vary smoothly'. The intuition behind the example comes from the nature of homogeneous degree one functions. Indifference sets from these functions are 'radially parallel' relative to the origin, and so one would expect this type of variation between indifference sets to be smooth. Theorem 1 states that this is so.

Theorem 1. Let $f\colon R^n_{++}\to R_+$ be a non-constant, continuous, homogeneous degree one function, and assume that all of its level sets are $n-1$ dimensional C^r manifolds. Then f is C^r at every point $x\in R^n_{++}$.

Proof. Choose x and let I_x be the level set containing x. By the definition of a manifold, there exists an open neighborhood U_x containing x and a C^r diffeomorphism $g_x\colon U_x\to R^n$ such that for some ($n-1$ dimensional) hyperplane H in R^n, $g_x^{-1}(H)=U_x\cap I_x$. We can assume that $\mathbf{0}$, the origin of R^n, is not in H (if it is, translate H away from the origin). There exists a continuous linear functional $L_x\colon R^n\to R$ with $L_x(\omega)=f(x)$ if and only if $\omega\in H$. Using L_x, it is possible to rewrite the function f over some neighborhood of x. Let $V_x\subset(U_x\cap I_x)\times\{t\,|\,|t-1|<\varepsilon\}$. If $t^*y^*=x$, then V_x is a neighborhood of (y^*,t^*). This comprises a change of coordinates for an open neighborhood of x, with the coordinates being $y\in I_x$ and $t\in R$. Using the definition $f(ty)\equiv L_x(t\cdot g_x(y))$, it can be seen that $f(ty)$ is C^r for $(y,t)\in V_x$. This establishes that f is C^r at x. Q.E.D.

If I is a C^r manifold in R^n_{++}, then $tI=\{tx\,|\,x\in I\}$ is a C^r manifold for all

[2]This example uses as consumption space the set $R^n_+-\{\mathbf{0}\}$, where $\mathbf{0}$ is the origin of R^n. Elsewhere in the paper attention is restricted to the open subset of this set, R^n_{++}. The axes were used in the example only to simplify the argument. If the same preference ordering were restricted to the open subset, it would still not be differentiable.

$t>0$.[3] So if I is a C^r manifold, then any homogeneous degree one function which has it as a level set must be a C^r function.

An application of Theorem 1 is that, given a smooth indifference manifold, one can construct a smooth function which has that indifference set as a level set. To do this, take the indifference set I, and construct a homogeneous degree one function from it. This idea is used in the next theorem.

Theorem 2. Let u be a utility function which represents the continuous, monotone, weakly C^r preference ordering $\geqq_p$ over R^n_{++}, and let I be an indifference set. Then there exists a C^r function v: $R^n_{++} \to R$ increasing in all its arguments with I as a level set.

Proof. For any $y \in R^n_{++}$, by continuity there exists a real number $v(y)>0$ and a point $w(y) \in I$ such that $v(y) \cdot w(y) = y$. By monotonicity, $v(y)$ and $w(y)$ are unique. Note that $tv(y) \cdot w(y) = ty$, so that v is a homogeneous degree one function. By Theorem 1, v is C^r, and by construction $v^{-1}(1) \equiv I$, and so I is a level set for v, which completes the proof. Q.E.D.

Any function v satisfying the requirements of Theorem 2 will be called a C^r *surrogate* function for the weakly C^r utility function u. The true utility function u will possess many different surrogate functions for each individual indifference set and different surrogate functions for different indifference sets. If preferences are smooth, so that u itself is smooth, then u can be its own surrogate function. Obviously, since any n-dimensional open manifold in R^n is diffeomorphic to R^n_{++}, surrogate functions could be constructed for preferences over other domains.

Given the indifference set I with $x \in I$, construct a smooth surrogate function v. Since v is smooth, it has a derivative at each point. In particular, it has a derivative at x. The derivative at x could be used to derive marginal rates of substitution for the function v. But, from the nature of the surrogate, the marginal rates of substitution for v at x give the slopes of the indifference set I at x, and these are exactly the marginal rates of substitution for u at x. The derivatives of the surrogate at points on the indifference set I serve as useful linear approximations to u, but are not derivatives of u.

To make this argument explicit, define the marginal rate of substitution between commodities i and j, MRS_{ij}, as the amount of commodity j which would just compensate the consumer for the loss of a marginal unit of commodity i [Hicks (1946, p. 20)]. If the utility function u is smooth, then $MRS_{ij} = u_i/u_j$, where u_i is the partial derivative of u with respect to commodity i. Marginal rates of substitution can also be thought of as slopes

[3]Suppose that for some $x \in I$, g is the local diffeomorphism with respect to the neighborhood U in the definition of manifold. To find a local diffeomorphism for tI at tx with respect to the neighborhood tU, take $t \cdot g(z/t)$ for $z \in tU$.

of indifference curves. If preferences are weakly smooth, then, indifference curves have slopes which can be used to derive marginal rates of substitution. This leads to the next theorem.

Theorem 3. If u is a utility function representing the continuous, monotone, weakly C^r preference ordering $\geqq_p$ over R^n_{++}, then MRS_{ij} exists at each point in R^n_{++} and for each i, $j=1,\ldots,n$ with $i \neq j$.

Proof. For any point $x \in R^n_{++}$, let I_x be the indifference set containing x. Let v be any C^r surrogate function for u relative to the indifference set I_x. By Theorem 2 such a function exists. Since v is smooth, $MRS^v_{ij}(x)$ exists and is unique for each $x \in R^n_{++}$ and each $i, j=1,\ldots,n$ with $i \neq j$. By definition, $MRS^v_{ij}(x)$ is the amount of commodity j that would compensate a person with utility function v and initial consumption x for the loss of a marginal unit of commodity i, keeping utility constant. Since I_x is an indifference set for both u and v, and $x \in I_x$, $MRS^u_{ij}(x) \equiv MRS^v_{ij}(x)$. Q.E.D.

Theorem 3 demonstrates the usefulness of surrogate functions in analyzing weakly smooth preferences, and the existence of marginal rates of substitution provides weakly smooth preferences with one of the basic tools of economic analysis. In the next section the consumer's constrained maximization problem is considered using this tool.

4. Consumer demand with weakly smooth preferences

The usual consumer demand problem when a consumer with utility function u is faced with an n-dimensional price vector $p \gg \mathbf{0}$ and wealth $w > 0$ involves choosing $x_1,\ldots,x_n$ to maximize $u(x_1,\ldots,x_n)$ subject to the budget constraint $p \cdot x \leqq w$. In this work we have assumed that consumption space is R^n_{++}, and so the budget set will not be compact. To avoid problems of existence of an optimum, we will restrict attention to (p, w) pairs for which the solution to the consumer's maximization problem exists and is in R^n_{++}. This is similar in spirit to the assumption that the solution point x^* is a regular point of demand [see, for example, Mas-Colell (1985)]. A (p, w) pair that yields an interior solution will be called an *interior point of demand*. If preferences are monotone, then the consumer will choose x so that $p \cdot x = w$ [see, for example, Varian (1978)]. If preferences are continuous and strictly convex and the solution to the maximization problem exists, then it is unique [see, for example, Barten and Bohm (1982)].

Assume that $\geqq_p$ is a monotone, continuous, strictly convex preference ordering over X, and that (p, w) is an interior point of demand. Let $\phi(p, w)$ denote demand for the n commodities $x_1,\ldots,x_n$ given the strictly positive price vector p and positive wealth w. Note that ϕ is continuous in all arguments. If $\geqq_p$ is also C^r, that is, if $\geqq_p$ can be represented by a C^r utility function u, then $\phi(p,w)$ is given by the solution to the system of equations

$$MRS_{ij}(x) \equiv u_i(x)/u_j(x) = p_i/p_j \quad \text{for} \quad \text{all } i \neq j, \qquad (1)$$
$$p \cdot x = w.$$

By strict convexity the solution to this system will be a maximum. These first-order conditions provide the basis for finding the first-order conditions for consumers with weakly smooth preferences.

Before doing this, however, first consider the relationship between maximizing a utility function and maximizing a surrogate function for that utility function.

Theorem 4. Let u and v be two continuous, increasing, real-valued functions defined on some set $X \subset R^n$. Suppose that x^* maximizes $u(x)$ subject to the constraint $x \in B \subseteq X$, and define I^* to be the level set containing x^*. If I^* is a level set for the function v, then x^* also maximizes $v(x)$ subject to $x \in B$.

Proof. Suppose that x^* does not maximize $v(x)$ subject to $x \in B$, that is, that there is some point $y \in B$ such that $v(y) > v(x^*)$. Since both functions are increasing and since I^* is a level set for both, $\{x \mid u(x) \geqq u(x^*)\} = \{x \mid v(x) \geqq v(x^*)\}$, and furthermore, $\{x \mid u(x) > u(x^*)\} = \{x \mid v(x) > v(x^*)\}$. Therefore $u(y) > u(x^*)$, which contradicts the assumption that x^* maximizes $u(x)$ subject to $x \in B$. Q.E.D.

Theorem 4 states that if some utility function chooses x^*, and if a surrogate function is constructed relative to the indifference set containing x^*, then the surrogate will also choose x^*. This allows one to use the first-order conditions from a smooth surrogate to characterize the optimum for a weakly smooth preference ordering.

Theorem 5. Let $\geqq_p$ be a monotone, continuous, strictly convex, weakly C^r preference ordering and let u be a utility function representing $\geqq_p$. Let $(p,w) \gg \mathbf{0}$ be an interior point of demand. If x^* maximizes $u(x)$ subject to $p \cdot x = w$, then x^* is the solution to the system of equations

$$MRS_{ij}(x) = p_i/p_j \quad \text{for} \quad \text{all } i \neq j \qquad (2)$$
$$p \cdot x = w.$$

Proof. Let I^* be the indifference set containing x^*. Let v be a homogeneous degree one C^r surrogate function relative to I^* as in the proof of Theorem 2.

Since v is homogeneous degree one and u is strictly quasiconcave and increasing, v is strictly quasiconcave and increasing. By Theorem 4, x^* maximizes $v(x)$ subject to $p \cdot x = w$. Therefore x^* satisfies the system of equations

$$MRS_{ij}^{v}(x) \equiv v_i(x)/v_j(x) = p_i/p_j \quad \text{for} \quad \text{all } i \neq j, \qquad (3)$$
$$p \cdot x = w.$$

By the proof of Theorem 3, since v is a smooth surrogate function for u, $MRS_{ij}^{u}(x) \equiv MRS_{ij}^{v}(x)$ for all $x \in I^*$, so that x^* also satisfies (2). Q.E.D.

In two dimensions, if preferences are smooth, monotone and strictly convex, then the consumer chooses a commodity bundle where an indifference curve is tangent to the budget line. If preferences are instead weakly smooth (and continuous), then indifference curves still have tangent lines, and so the consumer again chooses the point where the indifference curve is tangent to the budget line. The solution must satisfy conditions saying that the slope of the indifference curve is equal to the price ratio and that the consumption point must be on the budget line.

Once the first-order conditions for an optimum have been derived, it is common to do comparative statics analysis by differentiating the first-order conditions [Samuelson (1983)]. However, unless preferences are at least C^2, this might be impossible [for example, Debreu (1972, 1976a) and Katzner (1968)]. In spite of this, it is still useful to examine the comparative statics problem for smooth preferences which has as its solution the Slutsky equations. These separate the effect of a price change on demand into a substitution effect from a compensated price change and an income effect. If preferences are weakly smooth then the substitution effect derivative is well defined but the income effect derivative might not exist, since it necessarily implies movement from one indifference set to another. This is the subject of the next section.

5. Hicksian demand functions

The duality approach to demand analysis has led to two types of demand functions. The Marshallian demand function is derived from the ordinary constrained utility maximization problem discussed in section 4. The Hicksian, or compensated, demand function is derived from the constrained expenditure minimization problem. Hicksian demand functions are given by the vector-valued function $h: R^n_{++} \times R \to R^n$ by $h(p, u_0) = \operatorname{argmin}_x \{p \cdot x \mid u(x) \geqq u_0\}$. If u is continuous and strictly quasiconcave then h is continuous and homogeneous degree zero in prices [see, for example, Barten and Bohm (1982)].

In this section it will be demonstrated that if preferences are weakly C^r, then the Hicksian demand functions are C^{r-1} in prices. The intuition behind this statement is that Hicksian demand functions are constructed using individual indifference sets, which are smooth if preferences are weakly smooth. The smoothness property of the indifference set can then be used to establish the smoothness property of the Hicksian demand function. Of course, the Hicksian demand function will not be smooth in u, in general, because u is not a smooth function.

Before proceeding it is useful to examine the smoothness results for ordinary Marshallian demand functions. For simplicity, denote the Marshallian demand function by $\phi(p,w)$. Debreu (1972) showed that if preferences are C^r, monotone, and strictly convex, then ϕ is C^{r-1} if and only if all indifference sets have everywhere non-zero Gaussian curvature [see also Katzner (1968) and Mas-Colell (1985)]. The assumption of non-zero Gaussian curvature of indifference sets will be needed to discuss smoothness properties of Hicksian demand functions, and so Gaussian curvature must be discussed at length.

The following construction of Gaussian curvature follows that in O'Neill (1966).[4] Suppose that I is a C^r manifold with $r \geqq 2$. For each $x \in I$, define $q(x)$ to be the unique unit normal vector to I at x pointing into the better-than set $\{y \in X \mid y \geqq_p x\}$. Note that $q(x)$ is orthogonal to the tangent plane to I at x. The uniqueness of $q(x)$, then, results from the uniqueness of the tangent plane, which in turn comes from the smoothness of I. The function $q(x)$ can be expressed as $q(x) = \sum q_i(x) e_i$ where e_i denotes the n-vector with one as the ith component and zeros as the other components. For a given direction z in R^n, let $z[g]$ denote the directional derivative of the function g in the direction z. Note that directional derivatives are n-dimensional vectors. For a given z, then, define $S_x(z) = \sum z[q_i(x)] e_i$. $S_x(z)$ maps R^n into R^n and tells how the unit normal vector to I at x changes when x is moved in the direction z. By O'Neill (1966, p. 191), $S_x(z)$ is a linear function of z, and so S_x can be considered as an $n \times n$ matrix. The Gaussian curvature of I at x, $K(x)$, is defined as the determinant of the matrix S_x. Note that it is independent of the direction z.[5]

One of the most appealing features of Gaussian curvature from the standpoint of this paper is that it depends only on the characteristics of each individual indifference set. If preferences are at least C^2, then other,

[4]O'Neill's construction is for the particular case of manifolds in R^3, but none of the steps taken here are specific to that case. For other constructions and discussions of Gaussian curvature, see Debreu (1972), Malinvaud (1972), or Mas-Colell (1985).

[5]It is also independent of whether the unit normals are chosen to point into the better-than set or the worse-than set [see O'Neill (1966, p. 203)].

equivalent conditions can be found, most notably strong convexity [see Katzner (1968), Malinvaud (1972), and Barten and Bohm (1982)]. If u is a strictly quasiconcave C^2 utility function, then it is said to be strongly quasiconcave if $z'u_{xx}z<0$ for every element of $\{z\in R^n | u_x \cdot z=0, z\neq 0\}$, where $u_{xx}(x)$ is the matrix of second partial derivatives of u, $u_x(x)$ is the vector of first partials, and z' is the transpose of the vector z. This assumption is equivalent to the assumption of diminishing marginal rate of substitution as well as the assumption of non-zero Gaussian curvature and strict quasiconcavity. If preferences can be represented by a strongly quasiconcave utility function, then the preference ordering is said to be strongly convex. Since strong convexity depends on second-derivative properties, however, it is a meaningless concept when preferences are only weakly smooth. An additional characterization of non-zero Gaussian curvature which will be useful in Theorem 6 below is that, if u is at least C^2, then all its indifference sets have non-zero Gaussian curvature at every point if and only if the bordered Hessian matrix H given by

$$H(x)=\begin{bmatrix} u_{xx}(x) & u_x(x) \\ u_x(x)' & 0 \end{bmatrix} \tag{4}$$

is non-singular [see Debreu (1972) or Mas-Colell (1985)], where $u_x(x)'$ is the transpose of the vector of first partials.

The assumption of non-zero Guassian curvature has some intuition behind it. If the Gaussian curvature of an indifference set is zero at some point, then the indifference set is said to be flat at that point. A manifold which is flat everywhere is not necessarily a hyperplane. An example of a surface which has a Gaussian curvature of zero at every point is a circular cylinder. Movements around the cylinder cause the unit normal vector to change direction, but movements from end to end do not. Gaussian curvature will be zero if there is any direction for which the surface is 'flat'.

If an indifference set is flat at a point, then there is some direction in which the slope of the indifference set does not change. When doing constrained maximization problems of the usual type, price changes in this direction would cause continuous changes in demand, but not differentiable changes in demand. It turns out that if the indifference surface is flat, the bordered Hessian matrix is not invertible so that the Implicit Function Theorem cannot be used to establish smoothness properties of demand functions.

This said, we turn our attention to the smoothness properties of Hicksian demand functions for weakly smooth preference orderings. As before, since the consumption set is open it is necessary to rule out corner solutions. It will be assumed that (p,u_0) is an *interior point of Hicksian demand*, that is, that the solution to the constrained minimization problem exists and lies in R^n_{++}. As before, if preferences are monotone, then the solution is on the indifference set, and if preferences are strictly convex, then the solution is unique.

Theorem 6. Let $\geqq_p$ *be a monotone, continuous, strictly convex, weakly* C^r *preference ordering over* R^n_{++} *and let* u *be a positive-valued utility function representing* $\geqq_p$*, and assume that all indifference sets have non-zero Gaussian curvature at each point. If* u_0 *is an attainable level of utility, then* $h(p,u_0)$ *is a* C^{r-1} *function of* p *for all* $p\in R^n_{++}$ *such that* (p,u_0) *is an interior point of Hicksian demand.*

Proof. $h(p,u_0)$ is the value of x which minimizes $p\cdot x$ subject to $u(x)\geqq u_0$. By Theorem 2 there exists a unique homogeneous degree one C^r surrogate function v constructed relative to the indifference set defined by $I_0=u^{-1}(u_0)$ such that $v(I_0)=u_0$.[6] Since v is homogeneous degree one and I_0 has non-zero Gaussian curvature at every point, all of the level sets of v have non-zero Gaussian curvature at every point. Since v is C^r, its bordered Hessian matrix, denoted $H_v(x)$, is non-singular for each x. Let p_0 be a particular price vector for which (p_0,u_0) is an interior point of Hicksian demand. Then $h(p_0,u_0)$ is also given by the value of x which minimizes $p_0\cdot x$ subject to $v(x)=u_0$. $x=h(p,u_0)$ can be obtained in a neighborhood of (p_0,u_0) as the solution in x and λ of the system

$$\begin{aligned} &p-\lambda v_x(x)=0,\\ &v(x)-u_0=0. \end{aligned} \tag{5}$$

By monotonicity $\lambda\neq 0$. By the Implicit Function Theorem[7] the function that assigns (x,λ) to (p,u_0) is C^{r-1} in p in a sufficiently small neighborhood of (p_0,u_0) if and only if the bordered Hessian matrix H_v is non-singular. This

[6]Homogeneous degree one utility representations of a given preference ordering are unique up to scalar multiplication. To see this, if $u(\cdot)$ is a representation, then $u(tx)=tu(x)$, and so if $f(u(\cdot))$ is another representation, $f(u(tx))=f(tu(x))$ and if $f(u(\cdot))$ is homogeneous degree one, then $f(u(tx))=tf(u(x))$, which means that $f(tz)=tf(z)$, which in turn means that $f(\cdot)$ can only be multiplication by a constant. Translating this to preferences, different homogeneous degree one utility representations are given by different utility-level labels to a specific indifference set. So a unique homogeneous degree one utility representation can be found which assigns a given value to the specific indifference set.

[7]See, for example, Guillemin and Pollack (1974).

is exactly the condition guaranteed by non-zero Gaussian curvature, and so h is C^{r-1} in prices. Q.E.D.

Theorem 6 can be used to find what further conditions on preferences are sufficient for a monotone, strictly convex preference ordering with indifference sets which have everywhere non-zero Gaussian curvature to generate smooth Hicksian demand functions. Debreu (1972) answered this question for Marshallian demand functions, and found that the missing condition is smoothness of the preference ordering. Theorem 6 shows, however, that smoothness is stronger than what is needed for Hicksian demand functions. For them it is sufficient for preferences to be weakly smooth. This should not be surprising because Hicksian demand functions are constructed relative to single indifference sets and Marshallian demand functions are constructed using the entire indifference map. Weakly smooth preferences only prescribe properties for single indifference sets, whereas smooth preferences require smoothness conditions on the entire indifference map.

One application of Theorem 6 is that if preferences are weakly smooth, then the Slutsky substitution matrix exists. It is the same as the substitution matrix for any smooth surrogate function constructed relative to the indifference set containing the optimal consumption point. This means that, when considering the effects of a price change, as in the Slutsky equation, the usual substitution effect can be found. However, the income effect derivative will not exist in general and the Marshallian demand function will not be smooth in prices. This is consistent with the notion that if preferences are weakly smooth, then comparative statics derivatives will not exist, except in special cases where the parameter shift leaves the individual indifferent. Substitution effects leave the individual indifferent, while income effects necessarily require movements off of the original indifference set.

If indifference sets are not smooth manifolds, then Hicksian demand functions may not be differentiable in prices. For example, consider the parametrized curve in R^2, $I=\{(x,y)|x<1,\ y=1/x^2\} \cup \{(x,y)|x\geqq 1,\ y=1/x^{1/2}\}$. This curve is a C^1 manifold at every point except $(1,1)$. To see this, notice that I is a combination of the graphs of $y=x^{-2}$ and $y=x^{-1/2}$. The first derivative of the former is $-2x^{-3}$, and the derivative of the latter is $-\frac{1}{2}x^{-3/2}$. The limit of the former function as x approaches 1 from below is -2, and the limit of the latter as x approaches 1 from above is $-\frac{1}{2}$. So I cannot be a C^1 manifold.

Now consider any preference ordering having I as an indifference curve. It is possible to consider orderings which are monotone and strictly convex. The Hicksian demand function for any such preference ordering with respect to the utility level associated with I will not be smooth for price pairs where either $p_x/p_y=\frac{1}{2}$ or $p_x/p_y=2$. For example, set $p_y=1$, and vary p_x. For $p_x<\frac{1}{2}$, the Hicksian demand function for good x, $h_x(p_x,1,I)=(2p_x)^{-2/3}$, for

$\frac{1}{2} \leqq p_x \leqq 2$, $h_x(p_x, 1, I) = 1$, and for $p_x > 2$, $h_x(p_x, 1, I) = (p_x/2)^{-1/3}$. This function is not differentiable at $p_x = \frac{1}{2}$ and $p_x = 2$. So Hicksian demand functions need not be smooth. Of course, it is possible to get smooth Hicksian demand functions from preferences which are not even weakly smooth. Leontief preferences provide an example.

6. Functional form for weakly smooth preferences

Since utility functions representing weakly smooth preferences have marginal rates of substitution, intuition suggests that they must be smooth in directions which leave the level of utility unchanged but not necessarily in directions which change the level of utility. This section formalizes this intuition that derivatives along indifference sets exist. The analysis is similar to that in Dekel (1986) where preferences under uncertainty which satisfy the betweenness axiom but not necessarily linearity or even smoothness were considered. There it was shown that the function representing preferences must satisfy an equation similar to the one in Theorem 7 below.

If u is a utility function representing weakly smooth preferences, then u is a smooth function of x as long as changes in x do not cause changes in the level of utility. So the utility function $u(x)$ could be rewritten as some other function w such that both x and the level of utility $u(x)$ enter as arguments. If x changes (infinitesimally) but $u(x)$ does not, then w should be smooth with respect to these changes. If changes in x cause $u(x)$ to change, then since u is not smooth, w should not be smooth with respect to these changes either. The next theorem formalizes this notion that utility can be represented by a function of the form $u(x) = w(x, u(x))$.

Theorem 7. Let u be a positive-valued utility function which represents the continuous, monotone, weakly C^r preference ordering $\geqq_p$ over R^n_{++}. Then there exists a continuous real-valued function $w: R^n_{++} \times R_+ \to R$ such that $w(x, v)$ is C^r in x for fixed v and such that $u(x)$ is the unique value of v which solves $w(x, v) = v$.

Proof. For each indifference set I there is a unique level of utility a such that $u^{-1}(a) = I$. Let v_a be the unique homogeneous degree one C^r surrogate function for u relative to the indifference set I such that $v_a(y) = a$ if and only if $y \in I$. Define $w(x, a) \equiv v_a(x)$. Note that if a is held constant at a_0, then $w(x, a_0)$ is just a homogeneous degree one surrogate function defined relative to the indifference set given by $u^{-1}(a_0)$, and this surrogate function is C^r in x by construction.

To show that w is continuous, consider the sequence $(x, a_n) \to (x_0, a_0)$. Define t_n so that $u(t_n x_n) = a_n$ and $u(t_0 x_0) = a_0$. By the continuity of u, $t_n \to t_0$. Letting v_n be the unique homogeneous degree one function corresponding to the indifference set $u^{-1}(a_n)$ discussed in the preceding paragraph, and v_0 the one for $u^{-1}(a_0)$, $v_n(t_n x_n)=a_n$ and $v_0(t_0 x_0)=a_0$ by construction. By first-degree homogeneity, $w(x_n, a_n)=a_n/t_n \to a_0/t_0 = w(x_0, a_0)$.

To see that $u(x)$ is the unique solution to $w(x,v)=v$, suppose that for a given x, v solves $w(x,v)=v$ and also $v' \neq v$ solves $w(x,v')=v'$. Let x' be a 'correct' value for v' in the sense that $x' \in u^{-1}(v') \neq u^{-1}(v)$. But then $w(x', v')= v' = w(x, v')$, which means that $x \sim x'$ since they are on the same indifference set for a surrogate function constructed for x'. This is a contradiction.

Q.E.D.

Note, however, that since u is not smooth in x, in general, w cannot be smooth in both x and u. For if w were smooth in both x and u, then the Implicit Function Theorem would imply that u could be represented as a smooth function of x, which would be a contradiction.

Theorem 7 states that u could be written as a function which depends on both x and the level of utility at x. Furthermore, if the level of utility is held constant, then the function is smooth in x. In the proof of the theorem a particular function with these properties was constructed. In general, however, there will be as many choices for w as there are surrogates, so w will not be unique, even for a particular utility representation of preferences. Our object, though, was to find a way to write the given utility representation in the required form, and this was accomplished by the theorem. The existence of some w is sufficient, and the particular choice of w is irrelevant.

For each value of u, then, the function $w(\cdot, u)$ is a smooth function of x. Of course, there are values of x such that $w(x,u) \neq u(x)$. Each value of u corresponds to a different indifference set, and so the functions $w(\cdot, u)$ could be thought of as smooth surrogate functions relative to the appropriate indifference sets. The purpose of the function w, then, is to group all of these surrogates into one unified function. Nevertheless, it is useful to note that for a particular value of u, the domain of x in $w(x,u)$ is *not* restricted to the indifference set associated with the utility level u.

The function w provides a good intuitive tool for presenting many of the ideas in previous sections. Preferences are weakly smooth but not smooth if they can be represented by functions which are smooth for fixed levels of utility but not across indifference sets. Smooth surrogate functions can be found by taking $w(\cdot, u)$ for fixed u. Linear approximations can be found by approximating w for fixed u. The marginal rate of substitution between commodities i and j can be found by increasing x_i infinitesimally and finding how much x_j must decrease to leave u (or w) unchanged. The first-order conditions for constrained maximization problems are simply the first-order conditions for the function $w(\cdot, u^*)$ where u^* is the optimal level of utility, that is, $u(x^*)$ where x^* is the optimal value of x. The function w could also be used to construct the Hicksian demand functions by holding the level of utility constant. Smoothness properties of $w(\cdot, u)$ for fixed u could then be used to determine smoothness properties of the Hicksian demand functions.

References

Barten, Anton P. and Volker Bohm, 1982, Consumer theory, in: Kenneth J. Arrow and Michael D. Intriligator, eds., Handbook of mathematical economics, Vol. II (North-Holland, Amsterdam) 381–429.

Billingsley, Patrick, 1986, Probability and measure (Wiley, New York) 2nd ed.

Debreu, Gerard, 1954, Representation of a preference ordering by a numerical function, in: R.M. Thrall, C.H. Coombs and R.L. Davis, eds., Decision processes (Wiley, New York) 159–165.

Debreu, Gerard, 1970, Economies with a finite set of equilibria, Econometrica 38, 387–392.

Debreu, Gerard, 1972, Smooth preferences, Econometrica 40, 603–615.

Debreu, Gerard, 1976a, Smooth preferences: A corrigendum, Econometrica 44, 831–832.

Debreu Gerard, 1976b, Regular differentiable economies, American Economic Review 66, 280–287.

Dekel, Eddie, 1986, An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom, Journal of Economic Theory 40, 304–318.

Guillemin, Victor and Alan Pollack, 1974, Differential topology (Prentice-Hall, Englewood Cliffs, NJ).

Hicks, John R., 1946, Value and capital (Oxford University Press, Oxford) 2nd ed.

Intriligator, Michael D., 1971, Mathematical optimization and economic theory (Prentice-Hall, Englewood Cliffs, NJ).

Katzner, Donald W., 1968, A note on the differentiability of consumer demand functions, Econometrica 36, 415–418.

Malinvaud, E., 1972, Prices for individual consumption, quantity indicators for collective consumption, Review of Economic Studies 39, 385–405.

Mas-Colell, Andreu, 1974, Continuous and smooth consumers: Approximation theorems, Journal of Economic Theory 8, 305–336.

Mas-Colell, Andreu, 1985, The theory of general economic equilibrium: A differentiable approach (Cambridge University Press, Cambridge).

O'Neill, Barrett, 1966, Elementary differential geometry (Academic Press, New York).

Samuelson, Paul A., 1983, Foundations of economic analysis (Harvard University Press, Cambridge, MA) enlarged ed.

Varian, Hal R., 1978, Microeconomic analysis (Norton, New York).