Ambiguity aversion and hedging

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Abstract

The literature on asset markets with ambiguity averse traders establishes that unless buyers and sellers completely disagree on the set of priors, no trade can occur. In this paper demand for the asset comes from a desire to use it to hedge against financial risk from other activities, and Pareto optimal trade can occur between ambiguity averse hedgers and ambiguity averse sellers even with common priors. As with the rest of the literature, no trade occurs between ambiguity averse sellers and ambiguity averse buyers with no hedging motivation.

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1 Introduction

Using a standard investment setting with ordinary financial assets, Dow and Werlang (1992) show that when traders are ambiguity averse there exists a "no-trade zone" of prices for which investors neither buy nor sell.[1] These investors solve typical portfolio choice problems, choosing between a riskless and an ambiguous asset to optimize their own welfare. Portfolio choice is not the only motivation for asset demand, though. An important class of investors purchases assets, primarily derivatives, in order to hedge against financial risks created by their core activities. If returns on assets are ambiguous, and if these potential hedgers are ambiguity averse, no-trade zones such as those found by Dow and Werlang would inhibit the use of financial derivatives to hedge against inherent risk because the no-trade zones raise the possibility that no sellers will offer instruments that hedgers will buy.

I show that, in constrast to Dow and Werlang's result, ambiguity aversion does not give rise to no-trade zones for assets used as hedging instruments. Specifically, when ambiguity aversion is modeled using Gilboa and Schmeidler's (1989) maxmin expected utility model with multiple priors, sellers are risk neutral but ambiguity averse, buyers (hedgers) are risk averse and ambiguity averse, and financial markets are competitive on the seller's side, buyers purchase fully-covering derivatives. Consequently, buyers are able

[1] Dow and Werlang (1992) show that when prices are given exogenously there exists a range of prices at which an ambiguity averse trader elects to take no position, neither buying the asset nor selling it short, contrary to the expected utility model in which there is only a single price at which no position is taken. Mukerji and Tallon (2001) extend this analysis to a market setting allowing for endogenous prices, and find that the no-trade result still holds.

to hedge fully against both risk and ambiguity. The reason for the reversal is as follows. In the traditional asset market studied by Dow and Werlang, buyers do well in states of nature in which the price is likely to rise and poorly in states in which the price is likely to fall, and the opposite holds for sellers. Ambiguity aversion is captured by overweighting the states in which the agent does poorly and underweighting those in which she does well, and so buyers tend to overweight states in which the price is likely to fall and sellers tend to overweight states in which the price is likely to rise.[2] With hedging, however, sellers do poorly in the same states of nature in which the hedger's core profit is low, and so in derivatives markets the states of nature are aligned for the two parties and the no-trade zone disappears.

The results allow for a characterization of the benefits of information that reduces the ambiguity. Sellers are unaffected by the information because their market is competitive, but ambiguity-reducing information leads to lower prices for the derivatives, which in turn make the hedgers better off.

The paper proceeds as follows. Section 2 describes the environment and Section 3 describes the buyers' and sellers' decisions. Section 4 contains the main result, that hedgers always purchase full coverage in equilibrium, and follows up with the welfare implications of better information. Section 5 offers some brief conclusions.

[2] Dow and Werlang's (1992) result uses Schmeidler's (1989) Choquet expected utility model of preferences, which is less general than the Gilboa-Schmeidler (1989) model with multiple priors. Epstein and Wang establish the Dow-Werlang results in the Gilboa-Schmeidler model. Billot et al. (2000), Epstein (2001), Kajii and Ui (2006), and Chambers and Melkonyan (2007) further examine no-trade results in the Gilboa-Schmeidler multiple priors setting. The primary result (e.g. Billot et al.) is that if the set of multiple priors held by buyers overlaps the set held by sellers, no trade can result. Also see Rigotti and Shannon (2005) for no-trade results in a slightly different setting.

2 The environment

A subset of agents earn profit (or income) $\pi(r)$ from some core activity, where r is the realization of a random variable $\tilde{r}$ and $\pi(\cdot)$ is strictly increasing. For example, the agents could be involved in the import/export business with r denoting the appropriate exchange rate. Let $s \in \mathbb{R}$ denote a state of the world, with lower values of s corresponding to more adverse conditions for earning income. In state s the value of r is governed by the distribution $F(r|s)$. Because lower values of s correspond to more adverse conditions for the core activity, an increase in s leads to a first-order stochastically dominating (FOSD) shift in $F(\cdot|s)$, so that $F(r|s_2) \leq F(r|s_1)$ when $s_2 > s_1$.

In the absence of information, the set of possible states is an interval $S^0 = [\underline{s}, \overline{s}]$. Information, however, can refine the set of possible states of nature. To capture this, let Σ be the set of all nonempty, closed intervals in S^0, including single-element intervals. A typical element is denoted S^I and termed an *information set*. When $S^I \subset S^J$ we say that S^I is *less ambiguous* than S^J, and when $S^I = \{s\}$ for some $s \in S^0$ we say that S^I is *unambiguous*.

A *hedging instrument* is a contract that pays $b(r)$ when the realization of the random variable is r, with $b'(r) \leq 0$ so that the instrument pays more when r is lower. They will be refered to on occasion in the sequel using the more abstract terms *asset* and *derivative*.

Agents are assumed to be ambiguity averse, which is captured using the maxmin expected utility model with multiple priors of Gilboa and Schmeidler (1989). Specifically, agent i possesses a von Neumann-Morgenstern utility function u_i defined over payoffs and, when the set of possible states is S, prefers payoff distribution F^* to distribution F if and only if

$$\min_{s \in S} \int u_i(x) dF^*(x|s) \geq \min_{s \in S} \int u_i(x) dF(x|s). \tag{1}$$

Thus, the agent evaluates the expected utility of a lottery according to the worst state of nature for that particular lottery.

Hedgers are agents who use the financial instrument to hedge against income risk in their core activities. They are assumed to be both risk and ambiguity averse, so that their von Neumann-Morgenstern utility functions are increasing and concave. Furthermore, all hedgers are assumed to be identical and to have the same information set as all other agents. They earn profit $\pi(r)$ from their core activities, and they also earn proceeds from any assets purchased, receiving the benefit payment $b(r)$ when the realization of the random variable is r and paying the purchase price p for the asset. To allow for the possibility of full coverage, set the benefit schedule so that it fully covers losses due to low realizations of r. Letting $\bar{\pi}$ denote the maximum achievable profit from the core activity, this entails restricting $b(r) = \bar{\pi} - \pi(r)$.

Sellers of derivatives are, in contrast, assumed to be ambiguity averse but risk neutral. I assume a continuum of sellers, so that competition drives the price of the derivative down to the level that makes the sellers indifferent between writing the hedging contract and staying out of the market. The twin assumptions of risk neutrality and perfect competition on the seller's side allow for a clear comparison with the equilibrium outcome in a setting without ambiguity. With risk averse hedgers and risk neutral sellers, Pareto optimality dictates full risk-sharing, so that hedgers bear no risk and sellers bear all of it. Perfect competition on the seller's side guarantees that the hedging instruments sell at the actuarially fair price, in which case hedgers purchase full coverage and the Pareto optimal outcome is achieved.

The market may also include *investors* who, like sellers, are ambiguity averse but risk neutral. Also like sellers but unlike hedgers, they do not receive earnings from non-financial activities correlated with $\tilde{r}$. Investors take the opposite position of sellers, and the same position as hedgers, buying the derivative for price p and receiving the benefit payment $b(r)$. The difference between investors and hedgers is the motivation behind asset demand.

3 Markets for hedging instruments

First consider the decisions of sellers. For any information set S^I, an ambiguity averse seller will offer the contract $b(r)$ if and only if the price exceeds the worst-case-scenario expected payout. The value to the seller of writing the contract in state s and selling it for price p is $p - \int b(r)F(r|s)$, and so the seller is willing to sell if and only if

$$p \geq \max_{s \in S^I} \int b(r) dF(r|s). \tag{2}$$

By construction $b(\cdot)$ is a decreasing function and increases in s lead to FOSD shifts in $F(\cdot|s)$. Consequently, $p(s) \equiv \int b(r)dF(r|s)$ is a decreasing function of s, and (2) is satisfied if and only if $p \geq p(s_*^I)$, where $s_*^I = \inf(S^I)$. Perfect competition on the seller's side guarantees that this inequality binds, so the market price for the contract $b(\cdot)$ in information set S^I is $P(S^I) = p(s_*^I)$, which is the actuarially fair value of the benefit payment stream in state s_*^I. The following proposition follows immediately.

Proposition 1 *If S^I is less ambiguous than S^J then $P(S^I) \leq P(S^J)$.*

Proposition 1 states that reductions in ambiguity reduce the competitive price of the asset. It holds because the seller's ordering of the desirability of states has the highest-numbered state the best and the lowest-numbered state the worst. Ambiguity aversion leads the seller to evaluate lotteries according to the least-favorable state, in this case the lowest-numbered one. Information sets are intervals, and so reductions in ambiguity remove states from the ends of the interval, most pertinently the lowest-numbered states.

The hedger chooses how many units k of the derivative to purchase. If the buyer pays price p in state s for k units of the asset her expected utility is given by

$$U(k|s, p) = \int u(\pi(r) + k[b(r) - p])dF(r|s), \tag{3}$$

where u is an increasing, concave function.

When the information set is a single state, so that $S^I = \{s\}$, the hedger chooses k to maximize (3). This yields the first-order condition

$$\int u'(\pi(r) + k[b(r) - p])[b(r) - p]dF(r|s) = 0. \tag{4}$$

The second-order sufficient condition is

$$\int u''(\pi(r) + k[b(r) - p])[b(r) - p]^2 dF(r|s) < 0 \tag{5}$$

which is satisfied when the hedger is strictly risk averse. Furthermore, if the price of the derivative is the competitive price $p(s)$, it can be shown that the hedger chooses full coverage, that is, $k = 1$.

Proposition 2 *When there is no ambiguity, in equilibrium the hedger purchases full coverage.*

Proof. When there is no ambiguity the information set contains a single state s. In equilibrium competition drives the price down to $p(s) = \int b(r)dF(r|s)$. When $k = 1$ the first-order condition (4) reduces to $u'(\bar{\pi} - p(s)) \int [b(r) - p(s)]dF(r|s) = 0$. ■

Proposition 2 states that when there is no ambiguity the hedger purchases enough of the asset to fully cover any losses induced on non-financial profit from low values of r.

The hedger is ambiguity averse, though, and chooses k to solve

$$\max_k \min_{s \in S^I} U(k|s, p). \tag{6}$$

The next section provides the main result regarding markets for hedging insturments under ambiguity.

4 Risk sharing under ambiguity

Because sellers are risk neutral and hedgers are risk averse, in the absence of ambiguity sellers should bear all the risk and hedgers should be fully covered. Proposition 2 shows that this happens when there is no ambiguity. The purpose of this section is to determine whether this occurs when there is ambiguity.

To that end, let $k(S^I)$ denote the value of k chosen by hedgers when the information set is S^I. A hedger purchases full coverage if $k(S^I) = 1$, because then her total income is nonstochastic at $\bar{\pi} - P(S^I)$. She purchases partial coverage when $k(S^I) < 1$, and purchases excessive coverage when $k(S^I) > 1$. We are interested in circumstances under which $k(S^I) = 1$, because in those circumstances the outcome is the Pareto optimal risk sharing arrangement. Accordingly, we say that an information set S^I generates a *full risk-sharing arrangement* if $k(S^I) = 1$.

The main result follows.

Theorem 1 *Every information set generates a full risk-sharing arrangement.*

Proof. First recall that $s_*^I = \inf(S^I)$ and the competitive price for the hedging instrument in information set S^I is $p(s_*^I) = \int b(r)dF(r|s_*^I)$. When the hedger purchases full coverage maxmin expected utility is $U(k|s)|_{k=1} = u(\bar{\pi} - p(s_*^I))$ which is independent of s. Consequently, under full coverage utility is the same in every state. The result follows if any other value of k reduces utility in some state. Looking at state s_*^I, differentiating $U(k|s_*^I)$

and evaluating at $k = 1$ yields

$$U'(k|s_*^I) = u'(\bar{\pi} - \bar{p}) \int [b(r) - \bar{p}] dF(r|s_*^I) = 0. \tag{7}$$

Risk aversion guarantees that the second-order sufficient condtion (5) is satisfied, so that (7) identifies a maximum. Thus, any other value of k results in a reduction of utility in state s_*^I, and so $k(S^I) = 1$. ■

Theorem 1 states that equilibrium behavior always involves trade between the hedgers and sellers, and that the hedgers purchase hedging instruments that provide full coverage of their core risk. This is a surprising result in light of the earlier literature on asset markets, in which there would be no trade in this setting.

To contrast the result of Theorem 1 with the rest of the literature on asset markets under ambiguity, now allow investors to enter the market. The expected value to the investor of purchasing the contract $b(\cdot)$ in state s for price p is $\int b(r)F(r|s) - p$, and so the investor is willing to buy if and only if

$$p \leq \min_{s \in S^I} \int b(r) dF(r|s). \tag{8}$$

Obviously, the condition under which an investor is willing to buy (8) is incompatible with the condition under which a seller is willing to write the contract (2), and the two can only agree to trade if there is a single state, that is, if the information set is completely unambiguous. With any ambiguity at all, investors and sellers do not trade. According to Theorem 1, though, hedgers and sellers do trade and, furthermore, they achieve the

Pareto efficient full risk-sharing arrangement.

Risk neutrality of the sellers was key to achieving the full risk-sharing arrangement, but not for achieving trade between hedgers and sellers in equilibrium. Risk aversion on the part of the sellers would entail raising $P(S^I)$ to satisfy $\int v(P(S^I) - b(r))dF(r|s_*^I) = 0$, where $v(\cdot)$ is the seller's concave von Neumann-Morgenstern utility function. The fact that increases in s lead to FOSD shifts in $F(\cdot|s)$ imply that s_*^I is still the worst state for the seller, no matter how risk averse he is. The higher price for the asset means that the hedger purchases less of it, but since she also views s_*^I as the worst state, as long as the seller is less risk averse than the hedger some trade takes place in order to share risk optimally.

One feature of the competitive equilibrium is that hedgers not only fully hedge against risk, but they also fully hedge against ambiguity. Thus, in this model and with these preferences, the optimal risk-sharing arrangement drives the ambiguity-sharing arrangement, and the risk neutral but ambiguity averse sellers bear all the risk and all the ambiguity.[3]

The model also allows for easy welfare analysis of improved information. A seller's maxmin expected utility in information set S^I is zero by construction, because sellers are risk neutral and sell for the competitive price in the most adverse state. A hedger's maxmin expected utility in information set S^I is nonstochastic and equal to $u(\bar{\pi} - P(S^I))$. As Proposition 1 shows, information that reduces ambiguity reduces the price of the hedging instrument, so information makes hedgers better off. In particular, if improved

[3]Bose, Ozdenoren, and Pape (2006) explore ambiguity-sharing in an auction setting, and find that the ambiguity averse seller can always increase revenue by switching to an auction providing full insurance.

forecasts of $\tilde{r}$ refine the information set from S^I to S^J, the hedger's maxmin expected utility increases from $u(\bar{\pi}-p(s_*^I))$ to $u(\bar{\pi}-p(s_*^J))$, and so improved forecasts increase hedger welfare only to the extent that they rule out the most adverse states. Improved forecasts that rule out the least-adverse states do nothing in this setting to either change the equilibrium price of hedging instruments or to increase hedger welfare.

5 Conclusion

This paper shows that Pareto optimal risk sharing can take place when agents are ambiguity averse and have the same sets of priors. Risk sharing is achieved through the trade of a financial instrument, and so the result contrasts with those of Dow and Werlang (1992) and their successors who established that trade occurs in asset markets with ambiguity averse sellers only when the two sides of the market have completely different sets of priors (that is, the sets of priors have null intersections). The reason for the contrasting result here is the hedging motivation for purchasing assets. When the instrument is used to hedge against risk from core activities, the sellers and hedgers agree on the ordering of the priors, in which case risk sharing can occur. On the other hand, when buyers do not have a hedging motivation for purchasing the instrument, no trade occurs.

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