Ambiguity Aversion: An Axiomatic Approach Using Second Order Probabilities

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Abstract

This paper presents an axiomatic characterization of preferences which allow ambiguity aversion. It is assumed that decision makers treat risky lotteries and ambiguous lotteries separately, with preferences governed by the von Neumann-Morgenstern axioms for risk and a variant of the Savage axioms for ambiguity. These axioms imply that the decision maker chooses among risks according to expected utility, forms subjective second order probabilities over possible risks, and chooses among ambiguous lotteries according to a modified version of subjective expected utility. Furthermore, the decision maker has two utility functions, one governing attitudes toward risk and one governing attitudes toward ambiguity. Conditions governing ambiguity aversion, comparative ambiguity aversion, and decreasing ambiguity aversion are derived, and are similar to the familiar Arrow-Pratt characterizations for risk aversion.

1 Introduction

One of the famous problems that highlights the difference between risk and ambiguity is the two-color Ellsberg problem (see Ellsberg, 1961). A decision maker is faced with two urns. The first urn contains 50 red and 50 yellow balls, and the second contains 100 balls but in an unknown mixture of red and yellow. The decision maker will be paid \$10 if she can draw a yellow ball, and she must choose which urn to draw from. The first earn generates a known payoff distribution, so it is *risky*, but the second urn generates an unknown payoff distribution, so it is *ambiguous*. Subjects systematically avoid the ambiguous urn in favor of the risky urn, thereby creating the need for a model of choice behavior which can accommodate the distinction between risk and ambiguity.

The subjective expected utility models of Savage (1954) and Anscombe and Aumann (1963) begin this process, but both require preference to exhibit ambiguity neutrality, and thus cannot accommodate the Ellsberg behavior. Researchers have used two methods to generalize the subjective expected utility model to allow for ambiguity aversion: nonadditive probabilities (Schmeidler, 1989; Gilboa, 1987; Sarin and Wakker, 1992) and second order probabilities (Hazen, 1987; Segal, 1987). The advantages of the nonadditive probability models are that they are based on axioms and that they can easily accommodate ambiguity aversion. An example of the latter occurs when the subjective "probability" of drawing a red ball from the second urn above is less than 1/2, and so is the "probability" of drawing a yellow ball. These models have also generated a nice application to portfolio decision (Dow and Werlang, 1992).[1] The biggest disadvantage of these models is that, since the "probabilities" do not necessarily sum to one, a different "expectation" mechanism must be used.[2] Models based on second order probabilities assume that when considering a problem in which the probability distribution is unknown, the individual imagines all of the possible probability distributions that could hold and forms a probability distribution over possible probability distributions.[3] One advantage of these models is that ambiguity attitudes can be discussed in much the same way as risk attitudes, since the model is very similar to subjective expected utility.

The purpose of this paper is to provide an axiomatic framework for models of ambiguity aversion using second order probabilities. The primitives for this model are the horse lotteries and roulette lotteries of Anscombe and Aumann, but the axioms are similar to Savage's axioms over acts.[4] The resulting model has three parts. First, the individual has a preference function governing choice toward risk, that is, choice over roulette lotteries, and this preference function is termed an A-preference function. Second, the individual has a subjective second order probability measure over possible roulette lotteries. Since each roulette lottery is evaluated by an A-preference function, this is the same as the individual having a subjective probability measure over A-preference values. Finally, the individual has a utility function over A-preference values, and chooses among horse lotteries to maximize the subjective expected utility of A-preference values. Ambiguity attitudes are governed by the shape of this last utility function.

One of the key assumptions of this model is that horse lotteries and roulette lotteries are treated separately by decision makers. Similar assumptions are

[1] For similar results using a model with second order probabilities, see Hazen and Lee (1991).

[2] Instead of subjective expected utility, these models use Choquet expected utility, which is similar to the rank-dependent expected utility models of behavior toward risk. See the three nonadditive probability papers cited above, or Quiggin (1982), Yaari (1987), or Chew, Karni, and Safra (1987).

[3] For the two-color Ellsberg problem, this entails forming a subjective probability distribution over the 101 possible combinations of red and yellow balls.

[4] Roulette lotteries are lotteries in which the probability distribution is known. Horse lotteries are lotteries in which the distribution of outcomes is unknown and the outcomes are roulette lotteries. Acts are lotteries in which the distribution of outcomes is unknown and the outcomes are (monetary) payoffs.

made by Segal (1987), Hazen (1987), and Sarin and Wakker (1992). In the framework used here, the individual faces a horse lottery whose outcomes are roulette lotteries, or risks. If the individual forms subjective probabilities over the possible roulette lotteries, then it is possible to reduce the compound horse/roulette lottery to a single subjective probability distribution. This reduction returns us to the Anscombe and Aumann subjective expected utility model, though, and therefore cannot accommodate ambiguity aversion. Throughout the paper, then, it is necessary to assume that compound horse/roulette lotteries are not reduced.[5] Because horse lotteries and roulette lotteries are treated separately by decision makers, risk attitudes and ambiguity attitudes can be different.[6]

This paper is very similar to a series of papers by Hazen (1987, 1989, and Hazen and Lee, 1991). He uses a different set of axioms to produce a more general preference function, which he calls subjective weighted linear utility. Its relation to the model proposed here is the same as the relation between Chew's (1983) weighted utility model and the expected utility model of decisions toward risk. Hazen (1989) shows how his model can be used to discuss ambiguity aversion, and Hazen and Lee (1991) discuss comparative ambiguity aversion and what they call increasing ambiguity aversion.[7] The main difference between the work presented here and Hazen's work is that here the subjective expected utility functional form is retained (albeit in a modified form), thus simplifying the extensions of results from the subjective expected utility literature to situations in which decision makers are ambiguity averse.

In accordance with this goal, Section 2 presents an axiomatic characterization of the model, and Section 3 provides a characterization of ambiguity aversion similar to Pratt's (1964) characterization of risk aversion, including a notion of comparative ambiguity aversion. A notion of decreasing (in wealth) ambiguity aversion is more problematic, and it is investigated in Section 4. A brief summary can be found in Section 5. Proofs are collected in the appendix.

2 Axioms

To model risk, let D denote the space of probability distributions over some bounded interval X. The set D is the set of risky alternatives, which Anscombe and Aumann (1963) refer to as *roulette lotteries*. To model ambiguity, let S be the set of states of the world, with generic element s. Let Σ be the set of all subsets of S, with generic element E, which is interpreted as an event.

[5] For more on reduction of compound lotteries, see Segal (1990).

[6] There is some experimental evidence suggesting that risk attitudes and ambiguity attitudes differ. See Cohen, Jaffray, and Said (1985), Curley, Yates, and Abrams (1986), and Hogarth and Einhorn (1990).

[7] Their notion of increasing ambiguity aversion involves the individual's response when the outcomes are left the same but the probabilities of favorable outcomes increase. This is different from the notion of increasing risk aversion, which involves changing both initial wealth and the final outcomes, but not the probabilities, and it is also different from the notion of decreasing ambiguity aversion discussed in Section 4 below.

Savage (1954) defines an *act* as a function from S to X. Anscombe and Aumann define a *horse lottery* ϕ as a function from S to D, that is, ϕ assigns a probability distribution to each state in S. The resolution of an act is an outcome in the payoff space, while the resolution of a horse lottery is a roulette lottery, which is a probability distribution over payoffs.

Schmeidler (1989) uses an axiomatic characterization of preferences over horse and roulette lotteries to construct a model which allows ambiguity aversion through nonadditive probabilities. Gilboa (1987) constructs a nonadditive probability model using axioms on preferences over acts, and Sarin and Wakker (1992) unify the two approaches. The approach used in this paper is similar to Schmeidler's because it is based on preferences over horse and roulette lotteries rather than acts, and it is similar to Sarin and Wakker's in that it treats preferences over the two types of lotteries separately.

Begin with behavior toward risk. Let F and G denote elements of D, and let $\succeq_A$ be a preference relation defined on D. The following axioms are standard for $\succeq_A$.

A1: (A-Ordering) – $\succeq_A$ is complete, reflexive, and transitive.

A2: (Continuity) – The sets $\{G \in D | G \succeq_A F\}$ and $\{G \in D | F \succeq_A G\}$ are closed.

A3: (Independence) – $F \succeq_A F'$ if and only if $\alpha F + (1-\alpha)G \succeq_A \alpha F' + (1-\alpha)G$ for all $G \in D$ and all $\alpha \in (0,1)$.

These axioms are the standard axioms for expected utility toward risk, and if axioms (A1) - (A3) hold there is an expected utility representation of $\succeq_A$ (see, for example, Fishburn, 1970). In particular, there exists a utility function $u : X \to \mathbb{R}$, unique up to affine transformations, such that $F \succeq_A G$ if and only if

$$\int u(x)dF(x) \geq \int u(x)dG(x). \tag{1}$$

The three axioms above are more than are needed to guarantee the existence of a functional representation of preferences. In fact, (A1) and (A2) are enough (Debreu, 1954). If a functional representation of $\succeq_A$ exists, let it be denoted by V, and let v be a generic value of V.

Turning now to ambiguity, let Φ denote the set of all horse lotteries defined over D, and let $\succeq_B$ denote the preference relation over Φ. The goal is to derive subjective probabilities over states from preferences over Φ, and therefore axioms similar to either the Savage axioms or the Anscombe-Aumann axioms are needed. The axioms that follow are similar in form to the Savage axioms as presented in Fishburn (1970), except that here the axioms apply to horse lotteries, and in Savage's work they apply to acts. In all cases, $\phi, \phi', \psi, \psi' \in \Phi$ are horse lotteries, $F, F', G, G' \in D$ are roulette lotteries, and $E, E', E_i \in \Sigma$ are events. The horse lottery Δ_F is the degenerate lottery which yields F in every state, that is, $\Delta_F(s) = F$ for all $s \in S$, and these lotteries are called *constant lotteries*. The set E^c is the complement of E in S, that is, $S \setminus E$. A set E is *null* if $\phi \sim_B \psi$ whenever $\phi(s) = \psi(s)$ for all $s \in E^c$. It is said that $\phi = \psi$

on E if $\phi(s) = \psi(s)$ for all $s \in E$. It is said that $\phi \succeq_B \psi$ *given* E if and only if $\phi' \succeq_B \psi'$ whenever $\phi(s) = \phi'(s)$ for $s \in E$, $\psi(s) = \psi'(s)$ for $s \in E$, and $\phi'(s) = \psi'(s)$ for all $s \in E^c$.

B1: (B-Ordering) – $\succeq_B$ is complete, reflexive, and transitive.

B2: (Sure-thing principle) – If $\phi = \phi'$ and $\psi = \psi'$ on E, and $\phi = \psi$ and $\phi' = \psi'$ on E^c, then $\phi \succeq_B \psi$ if and only if $\phi' \succeq_B \psi'$.

B3: (Eventwise monotonicity) – If E is not null and if $\phi = \Delta_F$ and $\psi = \Delta_G$ on E, then $\phi \succeq_B \psi$ given E if and only if $F \succeq_A G$.

B4: (Weak comparative probability) – Suppose that $F \succeq_A G$, $\phi = \Delta_F$ on E, $\phi = \Delta_G$ on E^c, $\psi = \Delta_F$ on E', and $\psi = \Delta_G$ on E'^c, and suppose that $F' \succeq_A G'$, $\phi' = \Delta_{F'}$ on E, $\phi' = \Delta_{G'}$ on E^c, $\psi' = \Delta_{F'}$ on E', and $\psi' = \Delta_{G'}$ on E'^c. Then $\phi \succeq_B \psi$ if and only if $\phi' \succeq_B \psi'$.

B5: (Nondegeneracy) – $F \succ_A G$ for some $F, G \in D$.

B6: (Small event continuity) – If $\phi \succ_B \psi$, for every $F \in D$ there is a finite partition of S such that for every E_i in the partition, if $\phi' = \Delta_F$ on E_i and $\phi' = \phi$ on E_i^c then $\phi' \succ_B \psi$, and if $\psi' = \Delta_F$ on E_i and $\psi' = \psi$ on E_i^c then $\phi \succ_B \psi'$.

B7: (Uniform monotonicity) – For all $E \in \Sigma$ and for all $F \in \psi(E)$, if $\phi \succeq_B \Delta_F$ given E, then $\phi \succeq_B \psi$ given E. If $\Delta_F \succeq_B \phi$ given E, then $\psi \succeq_B \phi$ given E.

These axioms are the standard Savage axioms modified so that they govern preferences over horse lotteries instead of preferences over acts. The main difference between these axioms and Savage's, then, is that here probability distributions in D replace outcomes in X. A second modification is that in Savage's axioms, the preference ordering over outcomes in X is induced by the preference ordering over constant acts. Here, the preference ordering over roulette lotteries in D is kept separate, and axiom (B3) states that the preference ordering over constant horse lotteries is consistent with the preference ordering over D.

The axioms on preferences over Φ do not impose any particular behavior on preferences over D. For example, consider the sure-thing principle (axiom (B2)). To apply this axiom to preference over D, suppose that the horse lotteries $\phi, \phi', \psi, \psi' \in \Phi$ yield the following probability distributions in states $E, E^c \in \Sigma$, where $F, G, H, I \in D$:

	E	E^c
ϕ	F	H
ϕ'	F	I
ψ	G	H
ψ'	G	I

The sure-thing principle states that $\phi \succeq_B \psi$ if and only if $\phi' \succeq_B \psi'$, that is, preferences only depend on states in which the two horse lotteries being considered have different outcomes. This has the same spirit as the independence axiom (axiom (A3)), but with one key difference. Here the mixture ϕ of F and H is *not* a probability mixture, and thus it is not in D. Consequently, axiom (B2) does not imply axiom (A3), and thus (B1)-(B7) do not imply linearity of preferences over D, as axioms (A1)-(A3) do. The only restriction that is made is that (B3) guarantees that preferences on constant horse lotteries are identical to preferences over the corresponding roulette lotteries.

The axioms governing $\succeq_A$ and $\succeq_B$ restrict the form of functions representing preferences. This can be seen in the next theorem and its corollary.

Theorem 1 *Assume that an individual has preferences $\succeq_A$ over D and preferences $\succeq_B$ over Φ. If axioms (A1)-(A2) on $\succeq_A$ and axioms (B1)-(B7)on $\succeq_B$ on Φ hold then there exists a function $V : D \to \mathbb{R}$, a probability measure $\mu : \Sigma \to [0,1]$, and a function $w : \mathbb{R} \to \mathbb{R}$ such that, for all $F, G \in D$, $F \succeq_A G$ if and only if $V(F) \geq V(G)$, and for all $\phi, \psi \in \Phi$, $\phi \succeq_B \psi$ if and only if*

$$\int w(V(\phi(s)))d\mu(s) \geq \int w(V(\psi(s)))d\mu(s). \tag{2}$$

Moreover, the function V is unique up to increasing transformations and for a given specification of V the function w is unique up to increasing affine transformations.

Corollary 2 *Assume that an individual has preferences $\succeq_A$ over D and preferences $\succeq_B$ over Φ. If axioms (A1)-(A3) on $\succeq_A$ and axioms (B1)-(B7)on $\succeq_B$ on Φ hold then there exists a function $u : X \to \mathbb{R}$, a probability measure $\mu : \Sigma \to [0,1]$, and a function $w : \mathbb{R} \to \mathbb{R}$ such that, for all $F, G \in D$, $F \succeq_A G$ if and only if*

$$\int u(x)dF(x) \geq \int u(x)dG(x), \tag{3}$$

and for all $\phi, \psi \in \Phi$, $\phi \succeq_B \psi$ if and only if

$$\int w\left(\int u(x)d(\phi(s)(x))\right)d\mu(s) \geq \int w\left(\int u(x)d(\psi(s)(x))\right)d\mu(s). \tag{4}$$

Moreover, the function u is unique up to increasing affine transformations and for a given specification of u the function w is unique up to increasing affine transformations.

To keep these functions straight, the function u is referred to as an *A-utility* function, V is referred to as an *A-preference* function, and w is referred to as a *B-utility* function. The A-utility function u can be used to describe attitudes toward risk, while the B-utility function w can be used to discuss attitudes toward ambiguity, as shown in the next section. Finally, the preference function in (4) is called an *SEU^2 preference function*, and preferences which satisfy (A1)-(A3) and (B1)-(B7) are called *SEU^2 preferences*.

The proofs of Theorem 1 and its corollary are fairly straightforward. Axioms (A1) and (A2) together are equivalent to the existence of a continuous

A-preference function V representing $\succeq_A$ (Debreu, 1954). Substituting the A-preference values $V(F)$ for F and the "acts" $V(\phi)$ for ϕ into axioms (B1)-(B7) yields the standard Savage axioms governing "acts" which are functions mapping states into A-preference outcomes.[8] Letting $Y \equiv V(D)$, these acts are mappings from S into $Y \subseteq \mathbb{R}$. Savage's theorem then states that (B1)-(B7) are equivalent to the existence of a probability measure μ and a B-utility function w satisfying (2). The addition of axiom (A3) makes the A-preference function linear in the probabilities (see, for example Fishburn, 1970), and so there exists an A-utility function u such that $\succeq_A$ is represented by (3).

Note that both von Neumann-Morgenstern's expected utility and Savage's subjective expected utility are special cases of (4). Expected utility holds when there is no ambiguity, that is, when μ places probability one on a single state. Subjective expected utility holds when all horse lotteries assign degenerate distributions in D to every state. More specifically, when $\phi(s)$ is a degenerate distribution for every $s \in S$ and every $\phi \in \Phi$, let $x(\phi, s)$ denote the outcome assigned probability one by $\phi(s)$, and $V(\phi(s))$ can be represented by $x(\phi, s)$. Preferences are then represented by $\int w(x(\phi, s)d\mu(s)$. In the terminology used here, this is the case of ambiguity but no risk.

3 Ambiguity aversion

Suppose that preferences satisfy axioms (A1)-(A2) and (B1)-(B7), so that they can be represented by the function $W(\phi)$ given by

$$W(\phi) = \int w(V(\phi(s)))d\mu(s). \tag{5}$$

W is referred to as the *B-preference* function. Letting $G(v; \phi, \mu, V)$ be the subjective probability distribution over A-preference values generated by the subjective probability measure μ, the horse lottery ϕ, and the A-preference function V, (5) can be rewritten

$$\Lambda(\gamma(\cdot; \phi, \mu, V) = \int w(v)dG(v; \phi, \mu, V). \tag{6}$$

The function Λ looks exactly like an expected utility preference function with the utility function w, and therefore ambiguity aversion and comparative ambiguity aversion can be analyzed in an analogous fashion to risk aversion and comparative risk aversion in expected utility theory.

An individual is risk averse if, given the choice between a payoff distribution in D and a degenerate distribution in D with the same mean, she prefers the degenerate distribution. In (6) ambiguity is modeled as a probability distribution over A-preference values, and there is no ambiguity when the distribution is de-

[8] The standard Savage axioms can be found in Savage (1954) or Fishburn (1970), for example.

generate.[9] Analogously to the case of risk, an individual is said to be *ambiguity averse* if, given the choice between a probability distribution over A-preference values and a degenerate distribution with the same mean, she prefers the degenerate distribution. This is equivalent tot he concavity of the B-utility function w, and it is also equivalent tot he individual being willing to pay a premium to avoid the ambiguity.[10] Here the premium takes the form of a reduction in the A-preference value v, which can arise from two different mechanisms, both of which have received attention in the experimental literature (see the survey by Camerer and Weber, 1992). One mechanism involves a monetary payment to avoid the ambiguity, and this is analogous to the usual risk premium from the expected utility literature (see, for example, Pratt, 1964), with one modification. The monetary payment must be chosen to generate the desired change in v, and different functional specifications of the A-preference function V result in different monetary premia. The second mechanism involves a reduction in the probability of a more favored payoff distribution (in D), and again the magnitude of the probability reduction depends on the specification of V.

When the decision maker has SEU^2 preferences, the monetary ambiguity premium can also be shown graphically, as in Figure 1. Suppose that the decision maker has initial wealth x_0 and faces a horse lottery with only two possible states. If the first state occurs she wins $\varepsilon > 0$ with probability .25 and loses ε with probability .75. If the second state occurs she wins ε with probability .75 and loses ε with probability .25. The first quadrant of Figure 1 shows the expected utility of these two roulette lotteries, and these are labeled EU_1 and EU_2, respectively. The second quadrant shows the individual's B-utility function, which exhibits ambiguity aversion. The B-preference value of the horse lottery lies somewhere on the segment between points P and Q in the figure. Assume, for the sake of argument, that the subjective probability of each state is .5. Given this, the subjective expected final wealth level is x_0. The B-preference value lies at the midpoint of segment PQ, and the individual is indifferent between facing the horse lottery and receiving A-preference value v^* for sure. This A-preference value corresponds to the individual making a certain payment of a, and a is the monetary ambiguity premium.

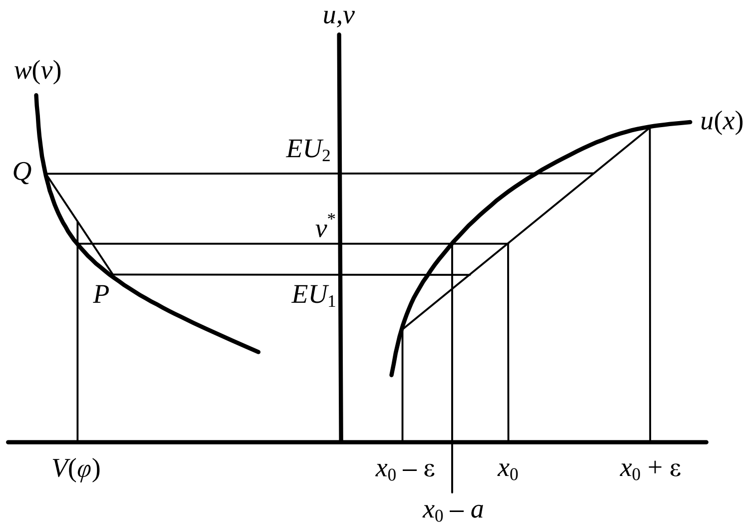
The fact that the size of the monetary and probability premia depend on the specification of V means that to compare the degrees of ambiguity aversion of two individuals, the two must have the same preferences over D, and the analysis must use the same A-preference function for both.[11] It is also necessary that the two individuals have the same subjective probability measure μ over states. Given these assumptions, a straightforward application of Pratt (1964, Theorem 1) establishes the equivalence of the following conditions, where it is assumed that w_1 and w_2 are both twice differentiable:

[9] Note that there may still be risk even when there is no ambiguity, since a degenerate distribution of A-preference values simply tells the A-preference value of the remaining roulette lottery, and not what the final payoff is.

[10] Similar results can be found in Hazen (1989) and Hazen and Lee (1991).

[11] This is a common assumption in comparative multivariate risk aversion. See, for example, Kihlstrom and Mirman (1974).

Figure 1



C1: $w_1(v) \equiv \rho(w_2(v))$ for some increasing concave function $\rho(\cdot)$.

C2: $-w_1''(v)/w_1'(v) \geq -w_2''(v)/w_2'(v)$ for all v.

C3: Let $z(\phi) \equiv \int v dG(v; \phi, \mu, V)$. If γ_1 and γ_2 solve $w_1(z(\phi) - \gamma_1) = \int w_1(v) dG(v; \phi, \mu, V)$ and $w_2(z(\phi) - \gamma_2) = \int w_2(v) dG(v; \phi, \mu, V)$, respectively, then $\gamma_1 \geq \gamma_2$.

Because they are equivalent, any of these three conditions can be used as a definition of "more ambiguity averse than." Consider (C3) first. The two individuals share the same preference ordering of D, and they have the same subjective probability measure μ, but they have different B-utility functions. An ambiguity premium is defined as the highest amount, measured in A-preference value units, that an individual would be willing to give up to avoid some noise in the distribution of A-preference values. Condition (C3) states that the more ambiguity averse individual is the one who is willing to forego more A-preference value units to eliminate ambiguity. Condition (C1) states that the more ambiguity averse individual's B-utility function is a concave transformation of the less ambiguity avers individual's, and (C2) states that the more ambiguity averse individual's B-utility function has a higher Arrow-Pratt measure than the less ambiguity averse individual's.

4 Decreasing ambiguity aversion

Many applications of the theory of risk use the notions of risk aversion and decreasing risk aversion. For example, Sandmo (1971) shows that a risk averse competitive firm facing a stochastic price variable reduces its output relative to the level of output when the price is nonstochastic. Furthermore, if the firm exhibits decreasing absolute risk aversion, a decrease in fixed cost causes an increase in output. These same issues arise when the distribution of prices is unknown. It is straightforward to show that an ambiguity averse firm reduces its output when the price changes from being nonstochastic to being stochastic but with an unknown distribution, and this result holds even if the ambiguity averse firm is risk neutral. The second problem, examining the impact of the change in fixed cost, requires a notion of decreasing ambiguity aversion.

Such a notion is problematic. Consider, for the moment, the case of a risk averse SEU^2 decision maker with initial wealth x_0 facing ambiguity ε. The individual forms a subjective distribution over possible risks ε_i, and each of these risks has an expected utility $EU(x_0 + \varepsilon_i)$. The individual then takes the subjective expected B-utility of these A-preference values. What happens if there is an increase in initial wealth to $x_1 > x_0$? The individual presumably forms the same subjective distribution over possible risks, but now the utility function is flatter, so the distribution of A-preference values is compressed. Even if the B-utility function w has a constant measure of ambiguity aversion, the individual is willing to pay less than before to avoid the ambiguity.

The problem arises because the A-preference function at initial wealth x_0 is different (locally) from the A-preference function at x_1, and comparisons of ambiguity aversion require that A-preferences be the same. Distributions of A-preference values are the same at two wealth levels only when the A-preference function is expected utility with a risk neutral utility function, otherwise there is either compression or decompression of the A-preference values.[12] Consequently, if the individual is risk neutral, it is possible to define notions of constant, decreasing, and increasing ambiguity aversion. To do so, abuse notation a little and let the distribution $\phi(s)$ denote the probability distribution of the change in wealth, and let $V(\phi(s), x)$ denote the A-preference value of the distribution $\phi(s)$ when initial wealth is x. Finally, let $G(v; \phi, x, \mu, V)$ denote the subjective probability distribution over A-preference values generated by the subjective probability measure μ, the horse lottery ϕ, the A-preference function V, and initial wealth x.

Proposition 3 *Assume that the individual has risk neutral SEU^2 preferences. Then the following conditions are equivalent:*

D1: $-w''(v)/w'(v)$ is a decreasing function of v.

D2: Let $z(\phi, x) \equiv \int v dG(v; \phi, x, \mu, V)$. If $\gamma(x)$ solves $w(z(\phi, x) - \gamma) = \int w(v) dG(v; \phi, x, \mu, V)$, then $\gamma(x)$ is decreasing.

[12] The same problem arises in the study of decreasing risk aversion in a multivariate setting, as investigated by Kihlstrom and Mirman (1981), with roughly the same results.

It is also possible to find corresponding conditions for constant and increasing ambiguity aversion.

The reason for assuming risk neutrality in Proposition 3 is to avoid compression of A-preference values when there is an increase in wealth. If we want to find decreasing ambiguity aversion, however, this compression works in the desired direction. Intuitively, decreasing ambiguity aversion means that as wealth increases the individual is willing to pay less to avoid a given ambiguity, and compression of A-preference values also reduces the amount the individual is willing to pay to avoid ambiguity. This yields another characterization of decreasing ambiguity aversion which does not rely on risk neutrality.

Proposition 4 *Assume that the individual has risk averse SEU^2 preferences. If $-w''(v)/w'(v)$ is a decreasing function of v, then condition (D2) holds.*

These two propositions establish (D2) as a definition of decreasing ambiguity aversion.

These notions of decreasing ambiguity aversion can be used to address problems like that of a competitive firm facing an ambiguous price. If the ambiguity averse firm exhibits decreasing ambiguity aversion under either of the above characterizations, then by a straightforward modification of Sandmo's proof it can be shown that a decrease in fixed cost results in an increase in output.

5 Conclusion

By using second order probabilities, and by treating horse lotteries and roulette lotteries separately, it is possible to characterize ambiguity aversion in much the same way as risk aversion. If an individual satisfies the usual expected utility axioms over roulette lotteries, then she behaves as an expected utility maximizer in situations of pure risk, and risk attitudes are governed by the shape of her von Neumann-Morgenstern utility function, which is termed an A-utility function in this paper. If, in addition, she satisfies a modified version of the Savage axioms over horse lotteries, then the individual forms a subjective probability distribution over states, and possesses another utility function, termed a B-utility function, which governs ambiguity attitudes. The existence of the B-utility function, and the fact that it enters the decision process in the same way as the von Neumann-Morgenstern utility function, makes it possible to extend many of the results from the study of situations of pure risk to situations of ambiguity.

A Appendix

Proof of Theorem 1. The existence of a function V which represents $\succeq_A$ is proved by Debreu (1954). Recall that a horse lottery ϕ assigns probability distributions in D to states in S. Define $\kappa_\phi(s) \equiv V(\phi(s))$, let K denote the set of all such κ_ϕ, and let $\succeq'_B$ be the preference ordering imposed on K by the preference ordering $\succeq_B$ over Φ. Axioms (B1)-(B7) imply the following axioms on $\succeq'_B$, where v is a real number, Δ_v is a constant lottery in K yielding preference value v, $Y \equiv V(D)$, and we use the notation that $\kappa_\phi = v$ on E if $\kappa_\phi(s) = v$ for all $s \in E$:

B1': $\succeq'_B$ is complete, reflexive, and transitive.

B2': If $\kappa_\phi = \kappa_{\phi'}$ and $\kappa_\psi = \kappa_{\psi'}$ on E, and $\kappa_\phi = \kappa_\psi$ and $\kappa_{\phi'} = \kappa_{\psi'}$ on E^c, then $\kappa_\phi \succeq'_B \kappa_\psi$ if and only if $\kappa_{\phi'} \succeq'_B \kappa_{\psi'}$.

B3': If E is not null and if $\kappa_\phi = v$ and $\kappa_\psi = v'$ on E, then $\kappa_\phi \succeq'_B \kappa_\psi$ given E if and only if $v \geq v'$.

B4': Suppose that $v_1 \geq v_2$, $\kappa_\phi = v_1$ on E, $\kappa_\phi = v_2$ on E^c, $\kappa_\psi = v_1$ on E', and $\kappa_\psi = v_2$ on E'^c, and suppose that $v_3 \geq v_4$, $\kappa_{\phi'} = v_3$ on E, $\kappa_{\phi'} = v_4$ on E^c, $\kappa_{\psi'} = v_3$ on E', and $\kappa_{\psi'} = v_4$ on E'^c. Then $\kappa_\phi \succeq'_B \kappa_\psi$ if and only if $\kappa_{\phi'} \succeq'_B \kappa_{\psi'}$.

B5': $v > v'$ for some $v, v' \in Y$.

B6': If $\kappa_\phi \succ'_B \kappa_\psi$, for every $v \in Y$ there is a finite partition of S such that for every E_i in the partition, if $\kappa_{\phi'} = v$ on E_i and $\kappa_{\phi'} = \kappa_\phi$ on E_i^c then $\kappa_{\phi'} \succ'_B \kappa_\psi$, and if $\kappa_{\psi'} = v$ on E_i and $\kappa_{\psi'} = \kappa_\psi$ on E_i^c then $\kappa_\phi \succ'_B \kappa_{\psi'}$.

B7': For all $E \in \Sigma$ and for all $v \in \kappa_\psi(E)$, if $\kappa_\phi \succeq'_B \Delta_v$ given E, then $\kappa_\phi \succeq'_B \kappa_\psi$ given E. If $\Delta_v \succeq'_B \kappa_\phi$ given E, then $\kappa_\psi \succeq'_B \kappa_\phi$ given E.

Axioms (B1')-(B7') are the usual Savage axioms for preferences on K, except with the implicit assumption that preferences are monotone over outcomes. By Savage's theorem (see, for example, Fishburn, 1970, Theorem 14.1), there exists a probability measure μ on Σ and a function $w : \mathbb{R} \to \mathbb{R}$, unique up to increasing affine transformations, such that $\kappa_\phi \succeq'_B \kappa_\psi$ if and only if

$$\int w(\kappa_\phi(s))d\mu(s) \geq \int w(\kappa_\psi(s))d\mu(s).$$

but $\kappa_\phi(s) \equiv V(\phi(s))$, which completes the proof.

Proof of Corollary 2. By Theorem 1 all that is left to prove is the linearity of V, which follows from the usual expected utility theorem (see, for example, Fishburn, 1970).

Proof of Proposition 3. Let η be the random variable given by $\eta(\phi, x, s) \equiv V(\phi(s), x) - z(\phi, x)$, so that $E[\eta] = 0$ when expectations are taken using the subjective probability measure μ. Since the individual's A-preferences are risk neutral, η is constant in x. The Proposition then follows from redefining (D2) in terms of z and η and applying Pratt (1964, Theorem 1).

Proof of Proposition 4. Assume that (D2) does not hold, that is, assume that there exists a horse lottery ϕ and initial wealth levels $x_0 < x_1$ such that $\gamma(x_0) < \gamma(x_1)$. Define $\eta(\phi, x, s) \equiv V(\phi(s), x) - z(\phi, x)$, so that $E[\eta] = 0$ when expectations are taken using the subjective probability measure μ. Since

$-w''(v)/w'(v)$ is a decreasing function of v, for $\gamma(x_0) < \gamma(x_1)$ it cannot be the case that $\eta(\phi, x_0, \cdot)$ is either less risky than (in the sense of Rothschild and Stiglitz, 1970) or the same as $\eta(\phi, x_1, \cdot)$. Then there exists a pair of states s_a and s_b in the support of μ such that

$$|V(\phi(s_a), x_0) - V(\phi(s_b), x_0)| < |V(\phi(s_a), x_1) - V(\phi(s_b), x_1)|.$$

But $V(\phi(s), x) = \int u(x+y)d(\phi(s)(y))$, so this implies that there exist outcomes y_a of $\phi(s_a)$ and y_b of $\phi(s_b)$ such that

$$|u(x_0 + y_a) - u(x_0 + y_b)| < |u(x_1 + y_a) - u(x_1 + y_b)|,$$

which contradicts the assumption that u is concave.

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