ASYMPTOTIC DIMENSION OF GROUPS

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Abstract. These notes are based on the paper [2].

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1. Ultrametric spaces

Definition 1.1. A metric space \((X, d)\) is called ultrametric if for all \(x, y, z \in X\) we have \(d(x, z) \leq \max\{d(x, y), d(y, z)\}\).

An ultrametric space \(X\) can be characterized by any of the following properties:

• **Isosceles triangles.** If a triangle in a space \(X\) has sides (distances between vertices) \(a \leq b \leq c\), then \(b = c\).

• **Radius \(\geq\) diameter.** For any ball its radius is greater or equal to its diameter.

• **Disjoint balls.** Two balls of radius \(D\) are either \(D\)-disjoint or identical.

Proposition 1.2. Let \((X, d)\) be a metric space. The metric \(d\) is an ultrametric if and only if \(f(d)\) is an ultrametric for every nondecreasing function \(f: \mathbb{R}_+ \to \mathbb{R}_+\).

**Proof.** If \(d\) is ultrametric and \(a \leq b = c\) are sides of a triangle in \((X, d)\) then \(f(a) \leq f(b) = f(c)\) are sides of the corresponding triangle in \((X, f(d))\) and therefore \(f(d)\) is an ultrametric.

If \(d\) is not an ultrametric then there is a triangle in \((X, d)\) with sides \(a \leq b < c\).

Consider the function

\[
 f(t) = \begin{cases} 
 t & \text{if } t \leq b \\
 \frac{2b}{c-b} t + \frac{bc-3b^2}{c-b} & \text{if } t \geq b 
\end{cases}
\]

The sides of the corresponding triangle in \((X, f(d))\) are \(f(a) \leq f(b) = b < 3b = f(c)\) which contradicts the triangle inequality. \(\square\)
Exercise 1.3. Let $d$ be the standard metric on the real line $\mathbb{R}$ and $q$ be a positive real number.

Is $d^q$ a metric on $\mathbb{R}$? [Not if $q > 1$]
If yes, what is the length of the interval $[0, 1]$ in $(\mathbb{R}, d^q)$?

Definition 1.4. A metric is said to be $10^n$-valued if the only positive values assumed by the metric are $10^n$, $n \in \mathbb{Z}$.

Exercise 1.5. Any $10^n$-valued metric is an ultrametric.

Exercise 1.6. Any $x^n$-valued metric is an ultrametric if $x > 2$.

Lemma 1.7. Any ultrametric space can be equipped with a $10^n$-valued ultrametric.

Proof. Apply Proposition 1.2 with the following function $f$:
$$f(d) = 10^n \quad \text{if} \quad 10^{n-1} < d \leq 10^n.$$ 

□

Exercise 1.8. Let $(X, d)$ be a metric space and $f$ is a nondecreasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$. Consider the identity map $F: (X, d) \to (X, f(d))$. When is $F$ continuous? [Answer: when $\lim_{t \to 0} f(t) = 0$] When is $F^{-1}$ continuous? [Always].

2. Universal ultrametric spaces

Let us introduce a new metric on the set $\mathbb{R}$ of real numbers. Given two numbers $a, b \in \mathbb{R}$, write them using decimal presentation:
$$a = a_ka_{k-1} \ldots a_0.a_{-1}a_{-2} \ldots$$
$$b = b_nb_{n-1} \ldots b_0.b_{-1}b_{-2} \ldots$$

Then look at these numbers from left to right and find the first index $m_{ab}$ where the presentations are different. Put
$$\mu(a, b) = 10^{m_{ab}}.$$

Exercise 2.1. Prove that $(\mathbb{R}, \mu)$ is an ultrametric space.

Let us describe an ultrametric space $(L_\omega, \mu)$ which is universal for all separable ultrametric spaces with $10^n$-valued metrics. This space appeared naturally in different areas of mathematics (see for example [3] and references therein). Let us fix a countable set $S$ with a distinguished element $s_0 \in S$. The set $L_\omega$ is a subset of the set of infinite sequences $\bar{x} = \{x_n\}_{n \in \mathbb{Z}}$ with all elements $x_n$ from the set $S$. A sequence $\bar{x}$ belongs to $L_\omega$ if there exists an index $k \in \mathbb{Z}$ such that $x_n = s_0$ for all $n > k$. The metric $\mu$ is defined as $\mu(\bar{x}, \bar{y}) = 10^m$ where $m \in \mathbb{Z}$ is the maximal index such that $x_m \neq y_m$. Clearly, the space $L_\omega$ is a complete separable $10^m$-valued ultrametric space.

To prove that any separable ultrametric space with $10^n$-valued metric embeds isometrically into $(L_\omega, \mu)$ we follow the idea of P.S. Urysohn [4] and show that the space $L_\omega$ is finitely injective:

Lemma 2.2. Let $(X, d)$ be a finite metric space with $10^n$-valued metric $d$. For any subspace $A \subset X$, any isometric map $f: A \to L_\omega$ admits an isometric extension $\tilde{f}: X \to L_\omega$. 

Proof. It is sufficient to prove Lemma in case $X \setminus A$ consists of one point $x$. In such case we have to find a point $\tilde{z} \in L_\omega$ such that $\mu(\tilde{z}, f(a)) = d(x, a)$ for every point $a \in A$. Let $A_x = \{a \in A | d(x, a) = d(x, A)\}$ be the set of all points in $A$ closest to $x$ and let $d(x, A) = 10^n$. Fix a point $b \in A_x$ and define $\tilde{z} = \{z_n\}_{n \in \mathbb{Z}}$ as follows: $z_m = f(b)_m$ if $m > n$; $z_m = s_0$ if $m < n$; $z_n$ is any element of the set $S$ other than $f(c)_n$ for any point $c \in A_x$.

Clearly, $\mu(\tilde{z}, f(c)) = 10^n = d(x, c)$ for any point $c \in A_x$. For any point $a \in A \setminus A_x$ we have $d(a, x) = d(a, b) = 10^n > 10^{m}$ which means that $f(a)_m \neq f(b)_m = z_m$ and therefore $\mu(\tilde{z}, f(a)) = 10^n = d(x, a)$. □

Exercise 2.3. Is the space $L_\omega$ finitely homogeneous (i.e. any isometry $f : A \to B$ of finite subsets $A, B \subset L_\omega$ extends to a surjective isometry of the whole space $L_\omega$)?

Problem 2.4. Is the space $L_\omega$ countably homogeneous (i.e. any isometry $f : A \to B$ of countable subsets $A, B \subset L_\omega$ extends to a surjective isometry of the whole space $L_\omega$)?

Definition 2.5. A subset $S$ of a metric space $X$ is called dense if for any point $x \in X$ and any number $\varepsilon > 0$ there is a point $s_x \in S$ with $d(s_x, x) < \varepsilon$.

A metric space is called separable if it contains a dense countable subset.

Exercise 2.6. Prove that the space $L_\omega$ is separable.

Exercise 2.7. If $X$ is a separable metric space and $f : X \to Y$ is a continuous surjective map, then $Y$ is separable.

Theorem 2.8. Any separable metric space $(X, d)$ equipped with $10^n$-valued metric $d$ embeds isometrically into the space $(L_\omega, \mu)$.

Proof. Since $X$ is separable, it is sufficient to embed isometrically a countable dense subspace $A$ of $X$. One can embed such a subspace by induction using Lemma 2.2. □

Exercise 2.9. Let $A, A', B, B'$ be four points in an ultrametric space $X$. If $d(A, A') < d(A, B)$ and $d(B, B') < d(A, B)$, then $d(A', B') = d(A, B)$.

Proposition 2.10. If $(X, d)$ is a separable ultrametric space, then $d$ assumes only countably many different values.

Proof. If a metric space $X$ is countable, then $d$ assumes only countably many different values. Since $X$ is separable, it contains a dense countable subset $A$. Since $A$ is countable, then $d$ assumes only countably many different values on pairs of points from $A$.

Now we show that for any points $x_1, x_2 \in X$ there are points $a_1, a_2 \in A$ such that $d(x_1, x_2) = d(a_1, a_2)$. Since $A$ is dense in $X$, there are points $a_1, a_2 \in A$ such that $d(x_1, a_1) < d(x_1, x_2)$ and $d(x_2, a_2) < d(x_1, x_2)$. Then by Exercise 2.9 we have $d(x_1, x_2) = d(a_1, a_2)$. □

Definition 2.11. Let $M$ be a countable subset of positive reals. A metric is said to be $M$-valued if the only positive values assumed by the metric are those from $M$.

Problem 2.12. Let $M$ be a countable subset of positive reals.

- Does there exist a universal separable $M$-valued ultrametric space?
- Is this space finitely homogeneous?
- Is this space countably homogeneous?
3. Ultrametric spaces and Lipschitz maps

**Definition 3.1.** A map \( f : X \to Y \) is called a non-expansive if \( d_Y(f(x), f(z)) \leq d_X(x, z) \) for any points \( x, z \in X \).

**Theorem 3.2.** A metric space \( X \) is ultrametric if and only if every non-expansive map \( f \) of a closed subset \( A \) of \( X \) to a 0-dimensional sphere \( S^0 \) can be extended to a non-expansive map of \( X \) to \( S^0 \).

**Proof.** Denote by \( B \) and \( C \) the preimages of points of \( S^0 \) under the map \( f \). Clearly, \( A = B \cup C \). If one of the preimages, say \( B \), is empty, then \( f \) is a constant map which has a constant extension to \( X \).

Consider a point \( x \in X \setminus A \) and compare the distances \( \text{dist}(x, B) \) and \( \text{dist}(x, C) \). Define an extension \( f \) on \( x \) as follows:

\[
f(x) = \begin{cases} 
  f(B) & \text{if } \text{dist}(x, B) \leq \text{dist}(x, C) \\
  f(C) & \text{if } \text{dist}(x, B) > \text{dist}(x, C)
\end{cases}
\]

Let us check that the map \( f \) is non-expansive. Consider two points \( x, z \in X \). If \( f(x) = f(z) \) then clearly \( d_{S^0}(f(x), f(z)) = 0 \leq d_X(x, z) \). Suppose that \( f(x) = f(B) \) and \( f(z) = f(C) \). Then \( \text{dist}(x, B) \leq \text{dist}(x, C) \) and \( \text{dist}(z, B) > \text{dist}(z, C) \) by definition of \( f \).

There is a point \( c_z \in C \) such that \( d(z, c_z) < \text{dist}(z, B) \). Then for any point \( b \in B \) the triangle \( c_z, z, b \) is isosceles with \( d(z, c_z) < d(z, b) = d(c_z, b) \geq \text{dist}(B, C) \geq D \) where \( D \) denotes the diameter of the 0-sphere \( d_{S^0}(f(B), f(C)) \). Therefore, \( d(z, b) \geq D \). If \( d(x, z) < D \) then \( d(x, c_z) < D \) (look at the triangle \( x, z, c_z \)) and \( \text{dist}(x, B) \leq \text{dist}(x, C) \leq d(x, c_z) < D \). Choose a point \( b \in B \) such that \( d(x, b) < D \) and consider the triangle \( x, c_z, b \). Since \( d(x, c_z) < D \) and \( d(x, b) < D \), then \( d(c_z, b) < D \) which contradicts \( \text{dist}(B, C) \geq D \). \( \square \)

**Exercise 3.3.** Prove the “if” part of Theorem 3.2.

**Definition 3.4.** A map \( r : X \to X \) is called a retraction if \( r(x) = x \) for every point \( x \in r(X) \).

A subspace \( A \subset X \) is called a retract of \( X \) if there exists a retraction of \( X \) onto \( A \).

It would be nice to have the following

**Non-Theorem.** A metric space \( X \) is ultrametric if and only if every closed subset \( A \) of \( X \) is a non-expansive retract of \( X \).

**Exercise 3.5.** Give a simple proof of Theorem 3.2 using Non-Theorem.

**Exercise 3.6.** Prove the “if” part of Non-Theorem.

**Example 3.7.** Let \( X = \{x_n\}_{n=1}^\infty \) be a sequence of points. Define \( d(x_1, x_n) = 1 + \frac{1}{n} \) and \( d(x_m, x_n) = \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\} \) for any \( m, n > 1 \).

**Exercise 3.8.** Show that \( d \) is an ultrametric on \( X \).

**Exercise 3.9.** Show that there is no non-expansive retraction of \( X \) onto \( A = \{x_n\}_{n=1}^\infty \).

**Definition 3.10.** A subspace \( A \subset X \) is called an almost non-expansive retract of \( X \) if for any \( \lambda > 1 \) there exists a retraction \( r_\lambda \) of \( X \) onto \( A \) such that \( d(r_\lambda(x), r_\lambda(z)) \leq \lambda \cdot d(x, z) \) for any points \( x, z \in X \).
\textbf{Theorem 3.11.} A metric space $X$ is ultrametric if and only if every closed subset $A$ of $X$ is an almost non-expansive retract of $X$.

\textit{Proof.} Suppose that $X$ is an ultrametric space and $A \subset X$ is a closed subspace. If $\lambda > 1$ is given, choose a number $\delta > 1$ such that $\delta^2 < \lambda$.

Let us fix an arbitrary well-order $\prec$ on $X$. We define a retraction $r: X \to A$ as follows. For a point $x \in X$ we look at the nonempty set

$$A_x = \{a \in A \mid d(x, a) \leq \delta \cdot \text{dist}(x, A)\}$$

and put $r(x)$ to be the minimal point in the set $A_x$ with respect to the order $\prec$.

Let us show that the retraction $r$ is $\lambda$-Lipschitz. Assume that for some points $x, y \in X$ we have $d(r(x), r(y)) > \lambda \cdot d(x, y)$. Without loss of generality we may assume that $r(x) \prec r(y)$.

If $d(y, r(x)) \leq d(y, r(y))$, then $r(x) \in A_y$ and $r(x) \prec r(y)$ contradicts the choice of $r(y)$ to be the minimal point in the set $A_y$.

In case $d(y, r(x)) > d(y, r(y))$ we denote by $D$ the distance between $r(x)$ and $r(y)$ and notice that $d(y, r(x)) = d(r(x), r(y)) = D$ in the isosceles triangle $\{y, r(x), r(y)\}$. Since $D > d(x, y)$, we have $d(x, r(x)) = d(y, r(x)) = D$ in the isosceles triangle $\{x, y, r(x)\}$.

$$d(x, r(y)) \geq \text{dist}(x, A) \geq \frac{1}{\delta} \cdot d(x, r(x)) = \frac{D}{\delta} > \frac{D}{\lambda} > d(x, y)$$

Therefore $d(x, r(y)) = d(y, r(y))$ in the isosceles triangle $\{x, y, r(y)\}$. The point $r(x)$ does not belong to $A_y$ since $r(x) \prec r(y)$, thus $d(y, r(x)) = D > \delta \cdot \text{dist}(y, A)$. Then there exists a point $z \in A$ with $d(y, z) < \frac{D}{\delta}$.

$$d(y, z) \geq \text{dist}(y, A) \geq \frac{d(y, r(y))}{\delta} = \frac{d(x, r(y))}{\delta} \geq \frac{D}{\delta} > \frac{D}{\lambda} > d(x, y)$$

Then $d(x, z) = d(y, z)$ in the isosceles triangle $\{x, y, z\}$. Since $d(x, z) < d(x, r(x))$, we have $z \in A_x$, but $d(x, z) < \frac{D}{\delta} = \frac{d(x, r(x))}{\delta}$ contradicts the definition of $A_x$ (two points $a, a' \in A_x$ cannot satisfy $d(x, a) < \frac{d(x, a')}{\delta}$).

\textbf{Exercise 3.12.} Prove the "if" part of Theorem 3.11.

\textbf{Definition 3.13.} A map $f: X \to Y$ of metric spaces is called Lipschitz if there is a constant $\lambda > 0$ such that the inequality $d_Y(f(x), f(x')) \leq \lambda \cdot d_X(x, x')$ holds for all points $x, x' \in X$. $f$ is called $\lambda$-Lipschitz if we need to specify the constant $\lambda$. $f$ is called $\lambda$-bi-Lipschitz if both $f$ and $f^{-1}$ are $\lambda$-Lipschitz.

For any Lipschitz map $f$ we denote 

$$\text{Lip}(f) = \inf\{\lambda \mid f \text{ is } \lambda\text{-Lipschitz}\}$$

Notice that a Lipschitz map $f$ is $\text{Lip}(f)$-Lipschitz.

\textbf{Exercise 3.14.} Prove or disprove: if $f: X \to Y$ and $g: Y \to Z$ are Lipschitz maps, then $\text{Lip}(g \circ f) = \text{Lip}(f) \cdot \text{Lip}(g)$.

Notice that the identity map $\text{id}: (X, d) \to (X, f(d))$ in Lemma 1.7 is expanding and 10-Lipschitz, thus it is 10-bi-Lipschitz.

\textbf{Corollary 3.15.} Any separable ultrametric space admits 10-bi-Lipschitz embedding into the space $(L_\omega, \mu)$.

\textit{Proof.} Combine Lemma 1.7 and Theorem 2.8. \hfill $\Box$
4. ASSOUAD-NAGATA DIMENSION

**Definition 4.1.** Let $X$ be a metric space, $A$ be a subspace of $X$, and $S$ be a positive number.

- $A$ is $S$-bounded if for any points $x, x' \in A$ we have $d_X(x, x') \leq S$.
- An $S$-chain in $A$ is a sequence of points $x_1, \ldots, x_k$ in $A$ such that for every $i < k$ the set $\{x_i, x_{i+1}\}$ is $S$-bounded.
- $A$ is $S$-connected if for any points $x, x' \in A$ can be connected in $A$ by an $S$-chain.

Notice that any subset $A$ of $X$ is a union of its $S$-components (the maximal $S$-connected subsets of $A$). If $B$ and $B'$ are two $S$-components of the set $A$ then $B$ and $B'$ are $S$-disjoint. Intuitively, a metric space $X$ has dimension 0 at scale $S > 0$ if all $S$-components of $X$ are uniformly bounded.

**Definition 4.2.** A metric space $X$ has Assouad-Nagata dimension zero (notation $\dim_{AN}(X) \leq 0$) if there exists a constant $m \geq 1$, such that for any $S > 0$ all $S$-components of $X$ are $mS$-bounded.

**Exercise 4.3.** Let $\mathbb{R}$ be the real line with the standard metric. Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}$ such that $x_{n+1} > x_n$ and $x_{n+1} - x_n > x_n - x_{n-1}$. What does it mean for the space $X$ to have Assouad-Nagata dimension zero?

**Exercise 4.4.** Bi-Lipschitz maps preserve Assouad-Nagata dimension zero.

Ultrametric spaces are the best examples of metric spaces of Assouad-Nagata dimension zero. Indeed, for any positive number $D$ any $D$-component of an ultrametric space is a $D$-ball and therefore is $D$-bounded. Let us characterize spaces of Assouad-Nagata dimension 0 using ultrametrics.

**Theorem 4.5.** If a metric space $(X, d)$ has Assouad-Nagata dimension $\dim_{AN}(X) \leq 0$, then there is an ultrametric $\rho$ on $X$ such that the identity map $\text{id}: (X, d) \to (X, \rho)$ is bi-Lipschitz.

**Proof.** Suppose that for a number $m > 1$, all $S$-components of $X$ are $mS$-bounded. Consider two points $x, z \in X$ and put

$$S = \frac{d(x, z)}{2m}.$$ 

Then the points $x$ and $z$ belong to different $S$-components of $X$. Thus for any chain $x = x_0, x_1, \ldots, x_{k-1}, x_k = z$ we have

$$d(x, z) \leq 2m \cdot \max_{0 \leq i < k} \{d(x_i, x_{i+1})\}.$$ 

Now define $\rho(x, z)$ to be the infimum of $\max_{0 \leq i < k} \{d(x_i, x_{i+1})\}$ over all finite chains $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x = x_0$ and $x_k = z$. Clearly

$$\frac{1}{2m} \cdot d(x, z) \leq \rho(x, z) \leq d(x, z).$$

To see that $\rho$ is an ultrametric, take three points $x, y, z$ in $X$ and let $s$ be the infimum of all positive numbers $S$ such that all three points belong to one $S$-component of $X$. If all three points belong to one $s$-component or all three belong to different $s$-components, then $\rho(x, y) = \rho(x, z) = \rho(y, z) = s$. If the points $x$ and $y$ belong to one $s$-component which does not contain $z$, then $\rho(x, y) \leq s = \rho(x, z) = \rho(y, z)$. $\square$
Theorem 4.6. Any separable metric space of Assouad-Nagata dimension 0 admits a bi-Lipschitz embedding into the space \((L_\omega , \mu)\).

Proof. Apply Theorem 4.5 and Theorem 3.15. \(\square\)

Definition 4.7. A metric space \(X\) is called a Lipschitz extensor for a metric space \(Y\) if there exists a constant \(m > 0\) such that for any closed subspace \(A \subset X\) any Lipschitz map \(f : A \to Y\) extends to a Lipschitz map \(F : X \to Y\) with \(\text{Lip}(F) \leq m \times \text{Lip}(f)\). We call the space \(Y\) an \(m\times\)-Lipschitz extensor for \(X\) if we need to specify the constant \(m\).

Theorem 4.8. The following conditions are equivalent:

1. \(\dim_{\text{AN}}(X) \leq 0\);
2. there exists a number \(\lambda\) such that every closed subset of \(X\) is a \(\lambda\)-Lipschitz retract of \(X\);
3. there exists a number \(\lambda\) such that every metric space is a \(\lambda\times\)-Lipschitz extensor for \(X\);
4. the unit 0-sphere \(S^0\) is a Lipschitz extensor for \(X\).

Proof. (1) \(\implies\) (2) Theorem 4.5 allows us to find an ultrametric \(\rho\) on \(X\) which is bi-Lipschitz equivalent to \(d\). Application of Theorem 3.11 completes the proof.
(2) \(\implies\) (3) Given a closed subspace \(A \subset X\) and a Lipschitz map \(f : A \to Y\) to some metric space \(Y\) we fix a \(\lambda\)-Lipschitz retraction \(r : X \to A\). Then the composition \(f \circ r : X \to K\) has the Lipschitz constant bounded by \(\lambda \cdot \text{Lip}(f)\).
(3) \(\implies\) (4) Obvious.
(4) \(\implies\) (1) Let \(m \geq 1\) be a number such that any \(\lambda\)-Lipschitz map from any closed subspace \(A \subset X\) to \(S^0\) can be extended to \(m\lambda\)-Lipschitz map of \(X\). If an \(S\)-component of \(X\) is not \(mS\)-bounded, there are points \(z_0\) and \(z_1\) with \(d(z_0, z_1) > mS\) and an \(S\)-chain of points \(z_0 = x_0, x_1, \ldots, x_k = z_1\). Notice that the map \(f : \{z_0\} \cup \{z_1\} \to S^0\) defined as \(f(z_0) = 0\) and \(f(z_1) = 1\) is \(\frac{1}{S}\)-Lipschitz but any extension of this map to the chain is at least \(\frac{1}{S}\)-Lipschitz and cannot be \(\frac{m}{d(z_0, z_1)}\)-Lipschitz (since \(\frac{1}{S} > \frac{m}{d(z_0, z_1)}\)). \(\square\)

Exercise 4.9. Prove the implication (2) \(\implies\) (1) directly.

5. Locally finite countable groups

All groups considered in this Section are countable.

Definition 5.1. A metric \(d\) on a group \(G\) is called left invariant if \(d(x, y) = d(g \cdot x, g \cdot y)\) for any \(x, y, g \in G\). In particular, \(d(x, y) = d(e, x^{-1} \cdot y)\).

A left invariant metric \(d\) on a countable group \(G\) is proper if and only if every bounded subset of \((G, d)\) is finite. Thus a left invariant proper metric \(d\) on \(G\) is bounded from below.

What can one say about two different left invariant proper metrics \(d\) and \(\rho\) on the same group \(G\)?

Definition 5.2. We call a map \(f : X \to Y\) of metric spaces uniform if there is a function \(\delta_f : \mathbb{R}_+ \to \mathbb{R}_+\) with \(\lim_{t \to 0} \delta_f(t) = 0\) such that \(d_Y(f(x), f(x')) \leq \delta_f(d_X(x, x'))\) for all points \(x, x' \in X\). To specify the function \(\delta_f\) we sometimes say that the map \(f\) is \(\delta_f\)-uniform. A map \(f\) is called bi-uniform if both \(f\) and \(f^{-1}\) are uniform.
**Theorem 5.3.** If $d$ and $\rho$ are two different left invariant proper metrics on the same group $G$, then the metric spaces $(G, d)$ and $(G, \rho)$ are bi-uniformly equivalent.

**Proof.**

**Definition 5.4.** Let $X$ be a metric space. We say that $X$ has asymptotic dimension zero (notation $\text{asdim}(X) \leq 0$) if there is a function $D_X^K : \mathbb{R}_+ \to \mathbb{R}_+$ (called 0-dimensional control function) such that for any $S > 0$ every $S$-component of $X$ is $D_X^K(S)$-bounded.

**Definition 5.5.** A group $G$ is called locally finite if every finitely generated subgroup of $G$ is finite.

**Example 5.6.** Direct sum of finite groups is locally finite.

**Theorem 5.7.** A countable group $G$ equipped with a proper left invariant metric has asymptotic dimension zero if and only if $G$ is locally finite.

**Proof.**

**Exercise 5.8.** Bi-uniform maps preserve asymptotic dimension zero.

Let $G$ be a locally finite countable group. Let us describe a particularly simple way to define a proper left-invariant metric on $G$. Consider a filtration $L$ of $G$ by finite subgroups $L = \{1 \subset G_1 \subset G_2 \subset G_3 \ldots \}$ and define the metric $d_L$ associated to this filtration as:

$$d_L(x, y) = \min \{i \mid x^{-1}y \in G_i \}.$$ 

**Exercise 5.9.** Show that $d_L$ is an ultrametric.

**Lemma 5.10.** Suppose two groups $G$ and $H$ have filtrations by finite subgroups: $L = \{1 \subset G_1 \subset G_2 \subset G_3 \ldots \}$ of $G$ and $K = \{1 \subset H_1 \subset H_2 \subset H_3 \ldots \}$ of $H$. If the index $[G_i : G_{i-1}]$ is less than or equal to the index $[H_i : H_{i-1}]$ for all $i$, then $(G, d_L)$ admits an isometric embedding into $(H, d_K)$. Moreover, if $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all $i$ (equivalently, the cardinality of $G_i$ equals cardinality of $H_i$ for all $i$), then the groups $(G, d_L)$ and $(H, d_K)$ are isometric.

**Proof.** Put $a_i = [G_i : G_{i-1}]$ and $b_i = [H_i : H_{i-1}]$. Fix an injection $f_1 : G_1 \to H_1$ and assume injections $f_k : G_k \to H_k$ are known for $k \leq n$ such that the following two properties hold:

1. $f_i(x) = f_j(x)$ for $i < j$ and $x \in G_i$,
2. the injection $f_k : G_k \to H_k$ is isometric.

Pick an injection of the set of cosets $\{x \cdot G_n \}$ of $G_n \subset G_{n+1}$ into the set of cosets $\{y \cdot H_n \}$ of $H_n \subset H_{n+1}$. That amounts to picking representatives $x_1, \ldots, x_m$ ($m = a_{n+1} - 1$) of cosets of $G_n$ in $G_{n+1}$ and picking representatives $y_1, \ldots, y_l$ ($l = b_{n+1} - 1$) of cosets of $H_n$ in $H_{n+1}$. Make sure the injection takes $\{1 \cdot G_n \}$ to $\{1 \cdot H_n \}$. Now we extend $f_n$ to $f_{n+1} : G_{n+1} \to H_{n+1}$ as follows: if $x \in G_{n+1} \setminus G_n$, we represent $x$ as $x_k \cdot x'$ for some unique $k \leq m$ and we define $f_{n+1}(x)$ as $y_k \cdot f_n(x')$.

If $x$ and $z$ belong to different cosets of $G_n$ in $G_{n+1}$, then $f_{n+1}(x)$ and $f_{n+1}(z)$ belong to different cosets of $H_n$ in $H_{n+1}$ and $d_L(x, z) = n + 1 = d_L(f_{n+1}(x), f_{n+1}(z))$. If $x$ and $z$ belong to the same coset $x_k \cdot G_n$ of $G_n$ in $G_{n+1}$, then $x = x_k \cdot x', z = x_k \cdot z'$. Since $f_{n+1}(x) = y_k \cdot f_n(x')$, $f_{n+1}(z) = y_k \cdot f_n(z')$, and the map $f_n$ is isometry, then $d_L(x, z) = d_L(x', z') d_H(f_n(x'), f_n(z')) d_H(f_{n+1}(x), f_{n+1}(z))$. 

By pasting all $f_n$ we get an isometric injection $f: G \to H$. Notice that in case $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all $i$, the map $f$ is bijective and establishes an isometry between $(G, d_L)$ and $(H, d_R)$. □

**Lemma 5.11.** Given two locally finite groups $G$ and $H$ the following conditions are equivalent:

1. There are left-invariant proper metrics $d_G$ on $G$ and $d_H$ on $H$ such that $(G, d_G)$ is isometric to $(H, d_H)$.
2. There are filtrations by finite subgroups: $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \ldots\}$ of $G$ and $\mathcal{K} = \{1 \subset H_1 \subset H_2 \subset H_3 \ldots\}$ of $H$ such that the cardinality of $G_i$ equals cardinality of $H_i$ for all $i$.

**Proof.** In view of 5.10, it suffices to show $(1) \implies (2)$. Obviously, we may pick an isometry $f: G \to H$ such that $f(1_G) = 1_H$ (replace any $f$ by $f(1_G)^{-1} \cdot f$). Notice $f$ establishes bijectivity between $m$-component of $G$ containing $1_G$ and the $m$-component of $H$ containing $1_H$. Also, those components are subgroups of $G$ and $H$. Thus, define $G_1$ as 1-component of $G$ containing $1_G$ and, inductively, $G_{i+1}$ as $(\text{diam}(G_i) + i)$-component of $G$ containing $1_G$. □

**Main example.** If $G$ is a direct sum of cyclic groups $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$, we consider the ultrametric on $G$ associated to the filtration

$$\mathcal{L} = \{1 \subset \mathbb{Z}_{a_1} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \mathbb{Z}_{a_3} \subset \ldots\}$$

If we write elements of the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$, as $p = p_1 p_2 \ldots p_n$ where $p_j \in \mathbb{Z}_{a_j}$, and denote $|p| = n$ then the ultrametric $d_{\mathcal{L}}$ can be defined explicitly as

$$d_{\mathcal{L}}(p, q) = \begin{cases} \max\{|p|, |q|\} & \text{if } |p| \neq |q| \\ \max\{|i | p_i \neq q_i| & \text{if } |p| = |q| \end{cases}$$

**Theorem 5.12.** A locally finite countable group $G$ with a proper left invariant metric $d$ is bi-uniformly equivalent to a direct sum of cyclic groups.

**Proof.** Fix a filtration $\mathcal{L}$ of $G$ by finite subgroups $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \ldots\}$. Then $(G, d)$ is bi-uniformly equivalent to $(G, d_{\mathcal{L}})$ by 5.3. By 5.10, $(G, d_{\mathcal{L}})$ is isometric to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$ where $a_i = [G_i : G_{i-1}]$. □

**Definition 5.13.** Let $G$ be a countable locally finite group and $p$ be a prime number. We define a $p$-Sylow number of $G$ (finite or infinite) as follows:

$$|p\text{-Syl}|(G) = \sup \{p^n | p^n \text{ divides } |F|, F \text{ a finite subgroup of } G, n \in \mathbb{Z}\}$$

Notice that if the $p$-Sylow number of $G$ is finite, it is equal to the order of a $p$-Sylow subgroup of some finite subgroup of $G$. For an abelian torsion group $G$ the $p$-Sylow number of $G$ is equal to the order of the $p$-torsion subgroup of $G$.

**Theorem 5.14.** Two countable locally finite groups $G$ and $H$ with proper left invariant metrics are bi-uniformly equivalent if and only if, for every finite subgroup $F$ of $G$, there exists a finite subgroup $E$ of $H$ such that $|F|$ is a divisor of $|E|$, and, for every finite subgroup $E$ of $H$, there exists a finite subgroup $F$ of $G$ such that $|E|$ is a divisor of $|F|$.
Let $G$ and $H$ be countable direct sums of finite prime cyclic groups. Let $d_G$ and $d_H$ be proper left invariant metrics on $G$ and $H$. Then the metric spaces $(G,d_G)$ and $(H,d_H)$ are bi-uniformly equivalent if and only if the groups $G$ and $H$ are isomorphic.

**Theorem 5.16.** Let $G$ and $H$ be locally finite countable groups with proper left invariant metrics $d_G$ and $d_H$. The metric spaces $(G,d_G)$ and $(H,d_H)$ are bi-uniformly equivalent if and only if for every prime $p$ we have $|p\cdot\text{Syl}(G)| = |p\cdot\text{Syl}(H)|$.

**Proof.** Assume the metric spaces $(G,d_G)$ and $(H,d_H)$ are bi-uniformly equivalent. Our goal is to show that if $|p\cdot\text{Syl}(G)| \geq p^n$, then $|p\cdot\text{Syl}(H)| \geq p^n$. If there is a finite subgroup $F$ of $G$ such that $p^n$ divides $|F|$, then by 5.14 there is a subgroup $E$ of $H$ such that $p^n$ divides $|E|$. Thus $|p\cdot\text{Syl}(H)| \geq p^n$.

Now suppose $|p\cdot\text{Syl}(G)| = |p\cdot\text{Syl}(H)|$ for every prime $p$. By 5.14, it is enough to show that for every finite subgroup $F$ of $G$, there exists a finite subgroup $E$ of $H$ such that $|F|$ is a divisor of $|E|$. If $|F| = p_1^{a_1} \cdots p_k^{a_k}$ then $p_i^{a_i} \leq |p_i\cdot\text{Syl}(H)|$ for every $i$. For every $i$ find a subgroup $E_i$ of $H$ such that $p_i^{a_i}$ divides $|E_i|$. Let $E$ be a finite subgroup of $H$ containing all the groups $E_i$. Clearly, $|F|$ divides $|E|$.

6. **COARSE EQUIVALENCE OF GROUPS**

**Definition 6.1.** Metric spaces $X$ and $Y$ are said to be coarsely equivalent if there are uniform maps $f: X \to Y$ and $g: Y \to X$ and a constant $C > 0$ such that both compositions $f \circ g$ and $g \circ f$ are $C$-close to the identity maps.

Let $G$ and $H$ be countable locally finite groups. Using 5.12 one can show that if

$$\sum_{p\text{-prime}} |p\cdot\text{Syl}(G) - p\cdot\text{Syl}(H)| < \infty$$

then the groups $G$ and $H$ are coarsely equivalent. Is the converse true?

**Problem 6.2.** Classify countable abelian torsion groups up to coarse equivalence.

Let us suggest a program to answer 6.2. Notice that any abelian torsion group is coarsely equivalent to a direct sum of groups $\mathbb{Z}_p$ with $p$ being prime. Therefore the following groups are of importance: $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p^{\infty}$ (the infinite direct sum of copies of $\mathbb{Z}_p$) and $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p^{n(p)}$, where $n(p) \geq 1$ for each $p \in \mathcal{P}$, $\mathcal{P}$ being a subset of primes.

**Problem 6.3.** Suppose $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p^{n(p)}$ and $\bigoplus_{q \in \mathcal{Q}} \mathbb{Z}_q^{m(q)}$ are coarsely equivalent. Is the symmetric difference of $\mathcal{P}$ and $\mathcal{Q}$ finite? If so, does $n(p)$ equal $m(p)$ for all but finitely many $p$?

**Problem 6.4.** Suppose $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p^{\infty}$ and $\bigoplus_{q \in \mathcal{Q}} \mathbb{Z}_q^{\infty}$ are coarsely equivalent. Is $\mathcal{P} = \mathcal{Q}$?

Call two countable abelian torsion groups $G$ and $H$ virtually isometric if there are subgroups of finite index $G'$ of $G$ and $H'$ of $H$ such that $G'$ is isometric to $H'$ for some choice of proper and invariant metrics on $G'$ and $H'$. Notice virtually isometric groups are coarsely equivalent.

**Problem 6.5.** Suppose two countable abelian torsion groups $G$ and $H$ are coarsely equivalent. Are $G$ and $H$ virtually isometric?
REFERENCES


