PHY611; Adv. QM + Intro. QFT. Problem Set 5

Due Wed. 19 Oct 2005 at the beginning of class.
n.b. Use natural units $\hbar = c = 1$ in all problems.

1. Dirac field theory

The Dirac field in terms of creation and annihilation operators is

$$\psi(\vec{x},t) = \sum_{s=1,2} \int \frac{d^3p}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{E_{\vec{p}}}} \left(b_{\vec{p},s} u_{\vec{p},s} e^{-ip\cdot x} + d^\dagger_{\vec{p},s} v_{\vec{p},s} e^{ip\cdot x} \right) . \tag{1}$$

Some of the following Dirac matrix elements are zero. Identify which are zero and state why.

a) (*1 pts*)

$$\langle 0|\psi^\alpha(\vec{x},t)|0\rangle \tag{2}$$

b) (*1 pts*)

$$\langle 0|\psi^\alpha(\vec{x},t)\psi^\beta(\vec{y},t)|0\rangle \tag{3}$$

c) (*1 pts*)

$$\langle 0|\bar{\psi}(\vec{x},t)\psi(\vec{y},t)|0\rangle \tag{4}$$

d) (*1 pts*)

$$\langle e^-(\vec{k}',\lambda')|\bar{\psi}(\vec{x},t)\psi(\vec{x},t)|e^-(\vec{k},\lambda)\rangle \tag{5}$$

e) (*1 pts*)

$$\langle e^-(\vec{k}',\lambda')|\psi^\alpha(\vec{x},t)|0\rangle \tag{6}$$

A *final* free Hamiltonian (which we have already seen) is the free Dirac Hamiltonian

$$H = \int d^3x\, \psi^\dagger(\vec{x},t)\Big(- i\vec{\alpha}\cdot\vec{\nabla} + \beta m\Big)\psi(\vec{x},t) \ . \tag{7}$$

Using the fact that ψ satisfies the Dirac equation, this can also be written as

$$H = i\int d^3x\, \psi^\dagger(\vec{x},t)\dot{\psi}(\vec{x},t) \ . \tag{8}$$

f) (*5 pts*) Substitute the Dirac field expansion into the latter form and show that the free Dirac Hamiltonian in terms of creation and annihilation operators is

$$H = \sum_{s=1,2} \int d^3p\, E_{\vec{p}} \left(b^\dagger_{\vec{p},s} b_{\vec{p},s} - d_{\vec{p},s} d^\dagger_{\vec{p},s} \right) . \tag{9}$$

2. Electron and positron charges

The total charge operator in Dirac field theory is given by

$$Q = -e \int d^3x \bar{\psi} \gamma^0 \psi \tag{10}$$

which is the volume integral of the time component of the conserved vector electromagnetic current $j^\mu = -e\bar{\psi}\gamma^\mu\psi$.

a) (*6 pts*) Substitute the field expansion Eq.(1) into this expression, and show that the total electric charge operator (relative to the charge of the vacuum) can also be written as

$$Q - \langle 0|Q|0\rangle = -e \sum_{s=1,2} \int d^3p \left(b^\dagger_{\vec{p},s} b_{\vec{p},s} - d^\dagger_{\vec{p},s} d_{\vec{p},s} \right) . \tag{11}$$

b) (*4 pts*) By evaluating the commutators $[Q, b^\dagger_{\vec{p},s}]$ and $[Q, d^\dagger_{\vec{p},s}]$, show that the effect of the electron or positron creation operator is to change the electric charge of a state by $-e$ or $+e$ respectively.

3. Potential scattering of Dirac fermions I

Our result for the leading order scattering amplitude for a Dirac electron in an external potential involved the matrix element of two Dirac fields, which was claimed to be

$$\langle e^-(\vec{p}', s')|\psi^\dagger(\vec{x}, t)\psi(\vec{x}, t)|e^-(\vec{p}, s)\rangle = \frac{1}{(2\pi)^3} \frac{m}{\sqrt{E_p E_{p'}}} u^\dagger_{\vec{p}',s'} u_{\vec{p},s} \, e^{i(p'-p)\cdot x} \tag{12}$$

a) (*6 pts*) Calculate this matrix element explicitly and see if the claimed result is correct.

It was also claimed without proof that the Fourier transform of a Coulomb potential is

$$\int d^3x \, e^{-i\vec{q}\cdot\vec{x}} \left(-\frac{e}{4\pi r} \right) = -\frac{e}{\vec{q}^2} \tag{13}$$

b) (*4 pts*) Calculate this integral and check the r.h.s. You will need to multiply the $1/r$ form by a convergence factor of $e^{-\lambda r}$ before doing this integral, then take the limit $\lambda \to 0$.