PHY611; Adv. QM + Intro. QFT. Problem Set 4

Due Wed. 5 Oct 2005 at the beginning of class.
n.b. Use natural units $\hbar = c = 1$ in all problems.

1. Scalar field theory matrix elements

The scalar field in terms of creation and annihilation operators is

$$\phi(\vec{x}, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x} \right) . \tag{1}$$

Some of the following scalar field matrix elements are zero. Identify which and state why.

a) (*2 pts*)

$$\langle 0|\phi(\vec{x}, t)|0\rangle \tag{2}$$

b) (*2 pts*)

$$\langle 0|\phi(\vec{x}, t)\phi(\vec{y}, t)|0\rangle \tag{3}$$

c) (*2 pts*)

$$\langle 0|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)\ldots\phi(\vec{x}_{2n+1}, t)|0\rangle \tag{4}$$

d) (*2 pts*)

$$\langle \vec{k}'|\phi(\vec{x}, t)|\vec{k}\rangle \tag{5}$$

e) (*2 pts*)

$$\langle \vec{k}'|\phi(\vec{x}, t)\phi(\vec{x}', t)|\vec{k}\rangle \tag{6}$$

2. Momentum in the scalar field

The momentum density carried by the scalar field (analogous to the Poynting vector in electromagnetism) is given by $-\dot{\phi}(\vec{x}, t)\vec{\nabla}\phi(\vec{x}, t)$. (This is the $0, i$ component of a very impressive object called the "stress-energy tensor" $T_{\mu\nu}$, which can be formed from the Lagrangian density $\mathcal{L}$.) The operator that measures the total linear momentum in a scalar quantum field theory is therefore

$$\vec{\Pi} = -\int d^3x\, \dot{\phi}(\vec{x}, t)\vec{\nabla}\phi(\vec{x}, t) \tag{7}$$

a) (*6 pts*) Insert the field expansions for ϕ and $\dot{\phi}$ in terms of creation and annihilation operators in this expression, and show that this total momentum operator can be written as

$$\vec{\Pi} = \int d^3k\, \vec{k}\, a^\dagger_{\vec{k}}\, a_{\vec{k}} . \tag{8}$$

(You can set $t = 0$ after calculating $\dot{\phi}$ to simplify this derivation.)

b) (*4 pts*) Show that this operator applied to a single particle state does indeed give the three-momentum vector,

$$\vec{\Pi}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle \ . \tag{9}$$

3. Photon field and operators

The expansion of the transverse part of the free photon field in terms of creation and annihilation operators is

$$\vec{A}(\vec{x},t) = \sum_{\lambda=1,2} \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(a_{\vec{k},\lambda}\, \hat{\epsilon}_{\vec{k},\lambda}\, e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda}\, \hat{\epsilon}^*_{\vec{k},\lambda}\, e^{ik\cdot x} \right) . \tag{10}$$

a) (*5 pts*) Given that the Hamiltonian for the free electromagnetic field is

$$H = \frac{1}{2} \int d^3x \left(\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2 \right) , \tag{11}$$

show that it can be written in terms of creation and annihilation operators as

$$H = \sum_{\lambda=1,2} \int d^3k\, \omega_{\vec{k}} \left(a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{1}{2}\, \delta_{\vec{k}}(\vec{0}\,) \right) . \tag{12}$$

b) (*5 pts*) The momentum carried by the electromagnetic field is an integral of the "Poynting vector" $\vec{\mathrm{E}} \times \vec{\mathrm{B}}$,

$$\vec{\Pi} = \int d^3x\, \vec{\mathrm{E}} \times \vec{\mathrm{B}} \ . \tag{13}$$

Derive the form of the corresponding photon momentum operator analogous to the expression for H given in Eq.(12). Show that this momentum operator acting on a one-photon state gives the result

$$\vec{\Pi}\,|\vec{k},\lambda\rangle = \vec{k}\,|\vec{k},\lambda\rangle \ . \tag{14}$$