PHY611; Adv. QM + Intro. QFT. Problem Set 3

Due Wed. 28 Sep 2005 at the beginning of class.

n.b. Use natural units $\hbar = c = 1$ in all problems.

1. Scalar field annihilation and creation operators.

The expansion for a scalar field $\phi(\vec{x}, t)$ at fixed $t = 0$ introduced in class was

$$\phi(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{+i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right) . \tag{1}$$

The time dependence of this or any other operator in the Heisenberg picture is given by

$$-i \frac{\partial}{\partial t} \mathcal{O} = [H, \mathcal{O}] . \tag{2}$$

a) (*5 pts*) Assuming that the free scalar Hamiltonian is

$$H = \int d^3p \, \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}} + E_0 \tag{3}$$

and that

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta(\vec{k} - \vec{k}') , \tag{4}$$

show that the time dependent field operator is just

$$\phi(\vec{x}, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^\dagger_{\vec{k}} e^{+ik\cdot x} \right) \tag{5}$$

where the dot products are $k \cdot x = \omega_{\vec{k}} t - \vec{k} \cdot \vec{x}$.

b) (*5 pts*) Solve for the creation operator $a^\dagger_{\vec{p}}$ explicitly in terms of $\phi(\vec{x}, t)$ and $\dot{\phi}(\vec{x}, t)$ using the field expansion given above (Eq.5) and spatial Fourier transforms). From this result, show that the ETCR (equal time commutation relation)

$$[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i\, \delta(\vec{x} - \vec{y}\,) \tag{6}$$

implies

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta(\vec{k} - \vec{k}') \tag{7}$$

as was used above.

2. Scalar field Hamiltonian.

(*10 pts*) The form of the Hamiltonian actually derived in class for the free scalar field was

$$H = \int d^3x \Big(\frac{1}{2}\dot{\phi}(\vec{x},t)^2 + \frac{1}{2}\vec{\nabla}\phi(\vec{x},t)^2 + \frac{1}{2}m^2\phi(\vec{x},t)^2 \Big) \ . \tag{8}$$

Substitute the scalar field expansion in the Heisenberg picture (Eq.5) into this H and show that it can indeed be written in terms of creation and annihilation operators as

$$H = \int d^3p \, \omega_{\vec{p}} \, a^\dagger_{\vec{p}} a_{\vec{p}} + E_0 \ . \tag{9}$$

3. Scalar field operators.

Consider the generic scalar field operator

$$\mathcal{O} = \int d^3k \, f(\vec{k}\,) \, a^\dagger_{\vec{k}} a_{\vec{k}} \ . \tag{10}$$

a) (*3 pts*) What constraints are required on $f(\vec{k}\,)$ to guarantee that $\mathcal{O}$ commutes with the free scalar Hamiltonian?

b) (*3 pts*) Show that the eigenvalues of $\mathcal{O}$ are additive, in the sense that the eigenvalue of $\mathcal{O}$ on an n-particle state is $\sum_{i=1}^{n} f(\vec{k}_i)$.

c) (*4 pts*) You may be getting tired of diagonal quadratic operators. To cure this exhaustion, write a quadratic operator analogous to the above that replaces any scalar one-particle state $|\vec{p}\rangle$ by a boosted state $|\vec{p}+\vec{k}\rangle$. (The boost operator in QFT is closely related to this.)