Dirac field theory (our final case: φ, A^μ, ψ).

S_{Dirac} should give $(i\not{\partial}-m)\psi(\vec{x},t)=0$ from $\delta S=0$.

And it should be Lorentz invariant, like $\bar{\psi}\psi$.

Easiest soln

$$L=\bar{\psi}(i\not{\partial}-m)\psi$$

$$S=\int d^4x\ \bar{\psi}(i\not{\partial}-m)\psi$$

If we are allowed to vary ψ and $\psi^\dagger$ independently!?

Actually $i\psi^\dagger(\vec{x},t)$ is the canonical momentum $\pi(\vec{x},t)$ conjugate to ψ.

$$\pi^\alpha(\vec{x},t)=\frac{\delta}{\delta\dot{\psi}^\alpha(\vec{x},t)}S=\int d^4x'\frac{\delta}{\delta\dot{\psi}^\alpha(\vec{x},t)}\left[i\psi^{\dagger\beta}\dot{\psi}^\beta+i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi-m\bar{\psi}\psi\right]$$

implicit

$$=\int d^4x'\left[i\psi^{\dagger\beta}\,\delta(x-x')\,\delta^{\alpha\beta}\right]$$

$$=i\psi^{\dagger\alpha}(\vec{x},t)\ ;$$

one should really think of S as

$$S(\pi,\psi)=-i\int d^4x\ \pi^\alpha(x)(i\not{\partial}-m)^{\alpha\beta}\psi^\beta(x)$$

$\quad\ \ \downarrow\ \ \downarrow$

$\quad\ \ p\ \ \ q$

and independently $\dfrac{\delta S}{\delta\pi}=0$ and $\dfrac{\delta S}{\delta\psi}=0$.

Convention has these as $\psi^\dagger$ and ψ, treated independently.

Now

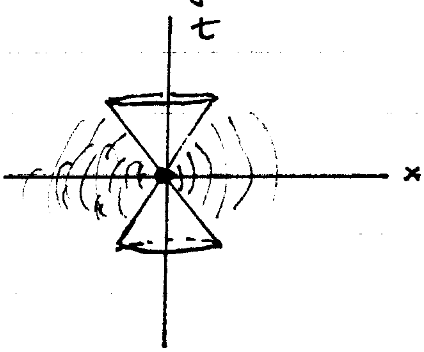
$$\frac{\delta S}{\delta \pi^\alpha(\vec{x},t)} = 0 = \left[(i\not\partial - m)\psi(\vec{x},t)\right]^\alpha \quad \checkmark \text{ Dirac equation}$$

Note $\bar\psi\psi$ and $\bar\psi\not\partial\psi = \bar\psi\gamma^\mu\frac{\partial}{\partial x^\mu}\psi$ are Lorentz invariant, so this S is as well.

Normally we impose canonical commutation relations (ETCR) between a field and it's canonical momentum, like

$$[\psi^\alpha(\vec{x},t), \pi^\beta(\vec{y},t)] = i\delta^{\alpha\beta}\delta(\overrightarrow{x-y}).$$

However in the Dirac case, if we do this we would <u>subsequently</u> find that the time-propagation of a localized disturbance would propagate <u>outside</u> the light cone

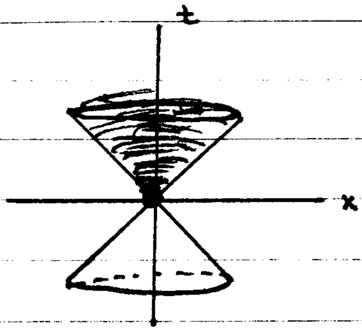


which violates causality.

To avoid this disaster, if one instead assumes <u>anticommutation</u> relations

$$\{\psi^\alpha(\vec{x},t), \pi^\beta(\vec{y},t)\} = i\delta^{\alpha\beta}\delta(\overrightarrow{x-y}) \qquad \{\psi,\psi\} = 0 = \{\pi,\pi\}$$

Then the response to a local source is causal,



Spin-statistics thm (Pauli, 1940).

Quantize bosons w/ comm S

fermions w/ anticomm. A

Implies multiparticle states are

One may think of this as implying that we are quantizing a classical field theory of <u>anticommuting</u> c-number fields, called Grassmann numbers

<u>classical</u>

$\varphi(\vec{x},t)$ (or $\vec{A},\vec{E}$)

$\mathring{\varphi}(\vec{x},t)$

$[\varphi(\vec{x},t),\varphi(\vec{y},t)]=0$

... (everything commutes)

<u>quantum</u>

$[\varphi(x,t),\mathring{\varphi}(y,t)]=i\hbar\,\delta(\vec{x-y})$

small correction to pretending φ is a c-number not an op.

but $[\varphi,\varphi]=0$

$[\mathring{\varphi},\mathring{\varphi}]=0$

still

$\psi(\vec{x},t)$

$\mathring{\psi}(\vec{x},t)$

$\{\psi(\vec{x},t),\psi(\vec{y},t)\}=0$

... (everything anticommutes)

$\{\psi(\vec{x},t),\pi(\vec{y},t)\}=i\hbar\,\delta(\vec{x-y})$

$=i\psi^\dagger$

but $\{\psi,\psi\}=0$

$\{\pi,\pi\}=0$

still.

In quantizing φ and A^μ we introduced Fourier expansions and identified $\pm$ ops $a, a^\dagger$, labels $\vec{k}$ or $\vec{k}, \lambda$.

These were identified by the comm rels with H,

$$[H, a^\dagger] = + \Delta E \; a^\dagger .$$

We can proceed similarly here, but there will be a "twist".

OK... we "Fourier transform" the Dirac field in a complete set of spinor · plane-wave modes (that satisfy Dirac eqn.)

$$u_{\vec{p}s}\, e^{-i\underbrace{p\cdot x}_{E_p t - \vec{p}\cdot\vec{x}}} \qquad\qquad v_{\vec{p}s}\, e^{+ip\cdot x}$$

$s = 1, 2$ $(\uparrow, \downarrow)$

$E_p = \sqrt{m^2 + \vec{p}^{\,2}}$

again time dep for the field operator is actually specified by the H_0 evolution predicted in the Heisenberg picture.

<u>Field expansion</u>

$$\psi(\vec{x}, t) = \sum_{s=1,2} \int d^3p \sqrt{\frac{m}{(2\pi)^3 E_p}} \left\{ B^{(u)}_{\vec{p}s} \underbrace{u_{\vec{p}s}\, e^{-ip\cdot x}}_{\substack{\text{plane wave}\\ \text{soln}\\ (+)E}} + B^{(v)}_{\vec{p}s} \underbrace{v_{\vec{p}s}\, e^{+ip\cdot x}}_{\substack{\text{p.w. soln}\\ (-)E}} \right\}$$

$B^{(u)}_{\vec{p}s}$, $B^{(v)}_{\vec{p}s}$ → operator coefts (temporary labels pending interpretation)

$$\psi^\dagger = \quad \cdots\cdots \quad B^{(u)\dagger}_{\vec{p}s}\, u^\dagger_{\vec{p}s}\, e^{+ip\cdot x} + B^{(v)\dagger}_{\vec{p}s}\, v^\dagger_{\vec{p}s}\, e^{-ip\cdot x} \Big\}$$

and, working out the commutator $\{\psi^{\alpha}(\vec{x},t), \psi^{\dagger\beta}(\vec{y},t)\} = \delta^{\alpha\beta}\delta(\overrightarrow{x-y})$

$$\{\psi, \psi\} = 0$$

we find

$$\left\{ B^{(u)}_{\vec{p},s}, B^{(u)\dagger}_{\vec{p}',s'} \right\} = \delta_{ss'}\delta(\overrightarrow{p-p}') = \left\{ B^{(v)}_{\vec{p},s}, B^{(v)\dagger}_{\vec{p}',s'} \right\}$$

all other combinations $\{\ ,\ \} = 0$.

To see what the $B, B^{\dagger}$ operators do, work out the Hamiltonian operator

$$H = \sum_i p_i \dot{q}_i - L$$

$$p_i \to i\psi^{\dagger}(x_i), \quad \dot{q}_i \to \dot{\psi}(x_i)$$

$$-\left[i\psi^{\dagger}\dot{\psi} + i\psi^{\dagger}(i\vec{\alpha}\cdot\nabla - \beta m)\psi \right]$$

$$= \psi^{\dagger}(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

$$H_{op} = \int d^3x\, \psi^{\dagger}_{op}(\vec{x},t)\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi_{op}(\vec{x},t)$$

$$= i\int d^3x\, \psi^{\dagger}_{op}(\vec{x},t)\, \dot{\psi}_{op}(\vec{x},t)$$

$$= \sum_{s=1,2}\int d^3p\, E_p \left(B^{(u)\dagger}_{\vec{p}s} B^{(u)}_{\vec{p}s} - B^{(v)\dagger}_{\vec{p}s} B^{(v)}_{\vec{p}s} \right)$$

"negative energy e^- states" made by $B^{(v)\dagger}_{\vec{p}s}$

Now we can work out commutators with H

$$\left[H_{op}, B^{(u)\dagger}_{\vec{p}s} \right] = +E_p B^{(u)\dagger}_{\vec{p}s}$$

$B^{(u)\dagger}_{\vec{p}s}$ creates a "fermion" with energy $E_p = +\sqrt{\vec{p}^2 + m^2}$

$$\left[H_{op}, B^{(v)\dagger}_{\vec{p}s} \right] = -E_p B^{(v)\dagger}_{\vec{p}s}$$

$B^{(v)\dagger}_{\vec{p}s}$ creates a fermion with energy $E_p = -\sqrt{\vec{p}^2 + m^2}$???

If so, the usual no-particle state $|0\rangle$

$$B^{(u)}_{\vec{p}s}|0\rangle = 0$$
$$B^{(v)}_{\vec{p}s}|0\rangle = 0$$

is not the lowest energy state.

$$B^{(v)\dagger}_{\vec{p}s}|0\rangle \quad \text{has lower energy than } |0\rangle, \text{ by } -\sqrt{\vec{p}^2+m^2}.$$

Similarly

$$\prod_{\text{all } \vec{p}} \prod_{s=1,2} B^{(v)\dagger}_{\vec{p}s}|0\rangle$$

$$\Big[\text{empty of } \underset{(+)\text{-Energy}}{u} \text{ quanta but } \underbrace{\text{totally occupied with } \underset{(-)\text{-Energy}}{v} \text{ quanta}}_{\text{"Dirac sea"}}\Big]$$

is the lowest energy state, $|\Psi_0\rangle$. The lowest-lying state is by defn the true vacuum

Starting from this $|\Psi_0\rangle$, we can either create e^- (+)-energy states with $E_p = \sqrt{\vec{p}^2+m^2}$

$$\boxed{B^{(u)\dagger}_{\vec{p}s}}|\Psi_0\rangle \longrightarrow \Delta E_{tot} = +\sqrt{\vec{p}^2+m^2}$$

normally called $b^\dagger_{\vec{p}s}$, e^- creation op $\quad \Delta Q = -e$ (add electron)

or we can remove a (-)-E e^- state. Net result is $\Delta E_{tot} = +\sqrt{\vec{p}^2+m^2}$

$$\boxed{B^{(v)}_{\vec{p}s}}|\Psi_0\rangle \longrightarrow \Delta E_{tot} = +\sqrt{\vec{p}^2+m^2}$$

$\Delta Q = +e$ (remove neg-E electron)

This acts just like creating a (+)-E electron of positive charge

normally called $d^\dagger_{\vec{p}s}$, e^+ creation op.

$$\{b_{\vec{p}s}, b^\dagger_{\vec{p}'s'}\} = \delta_{ss'}\,\delta(\vec{p}-\vec{p}')$$
$$= \{d_{\vec{p}s}, d^\dagger_{\vec{p}'s'}\}$$

other $\{,\} = 0$.

And we can just call $|\Psi_0\rangle$ "the vacuum" $|0\rangle$.

So, we expand ψ as

$$\psi(\vec{x},t) = \sum_{s=1,2} \int d^3p \sqrt{\frac{m}{(2\pi)^3 E_p}} \left\{ b_{\vec{p}s}\, u_{\vec{p}s}\, e^{-ip\cdot x} + d^\dagger_{\vec{p}s}\, v_{\vec{p}s}\, e^{ip\cdot x} \right\}$$

ψ annihilates an e^- or (and) creates an e^+. Changes the charge of a state by $+e$.

$$\psi^\dagger = \quad . \quad . \quad . \quad . \quad . \quad \left\{ b^\dagger_{\vec{p}s}\, u^\dagger_{\vec{p}s}\, e^{+ip\cdot x} + d_{\vec{p}s}\, v_{\vec{p}s}\, e^{-ip\cdot x} \right\}$$

$\psi^\dagger$ creates e^- and/or annihilates e^+. Changes the charge of a state by $-e$.

Some other operators $\qquad$ n.b. $j^\mu \equiv -\frac{\delta S}{\delta A^\mu}$, $\quad Q = \int d^3x\, j^0$

$$Q = -e\int d^3x\, \psi^\dagger \psi = -e\sum_s \int d^3p\, \left(b^\dagger_{\vec{p}s} b_{\vec{p}s} - d^\dagger_{\vec{p}s} d_{\vec{p}s} \right)$$

$$= -e\left(N^{(e^-)}_{op} - N^{(e^+)}_{op} \right)$$

$$\vec{P}_{op} = \int d^3p\, \vec{p}\, \left(b^\dagger_{\vec{p}s} b_{\vec{p}s} + d^\dagger_{\vec{p}s} d_{\vec{p}s} \right)$$

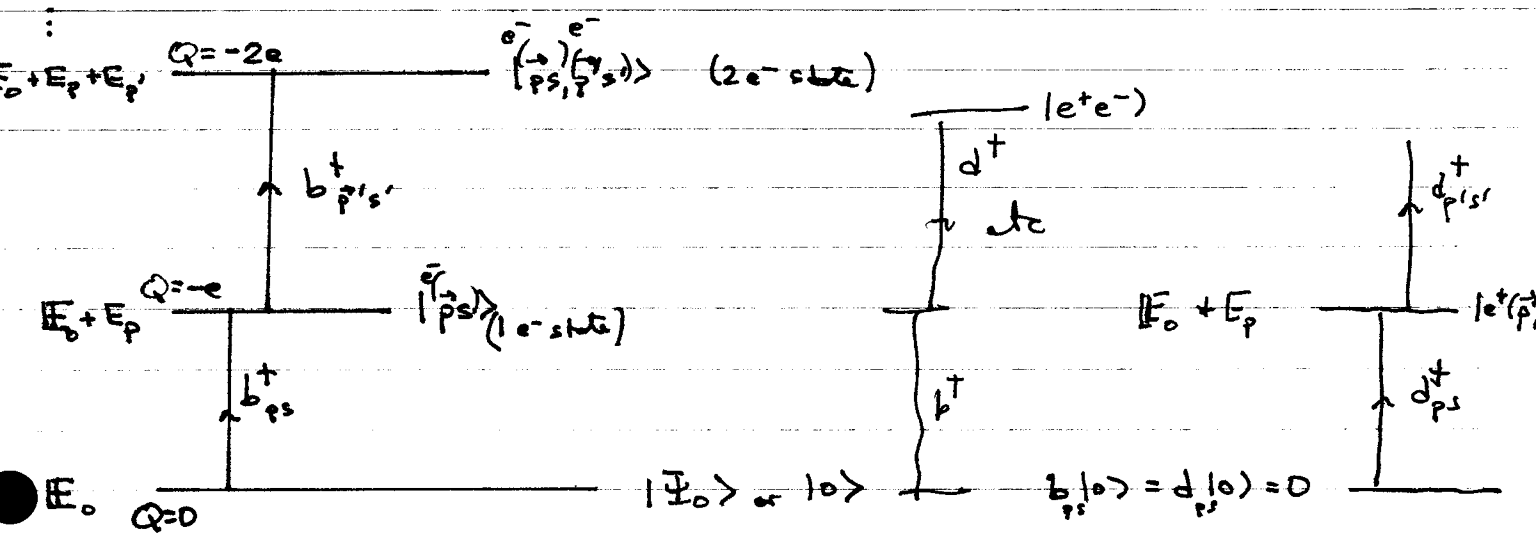
$$H_{op} = \int d^3p\, E_p \left(b^\dagger_{ps} b_{ps} + d^\dagger_{ps} d_{ps} \right) \underbrace{- 2\delta_p(\vec{0}) \int d^3p\, E_p}$$

E_0 of $|\Psi_0\rangle$ for Dirac fermion.

Negative zero point energy.

$(-4 \times E_0 \text{ of } \varphi)$

Hilbert space



Similarly with positrons

$|e^-(\vec p s)\rangle = b^\dagger_{ps}|0\rangle$ $\qquad$ $|e^+(\vec p, s)\rangle \equiv d^\dagger_{ps}|0\rangle$ one positron state

$Q = +e \qquad E_0 + E_p$

$|e^-(p,s)\, e^+(\bar p,\bar s)\rangle = b^\dagger_{ps}\, d^\dagger_{\bar p \bar s}|0\rangle$ one e^- + one e^+ etc.

$Q=0$ Watch order!

n.b. $\{b^\dagger_{ps}, b^\dagger_{p's'}\} = 0$ so $(b^\dagger_{\vec p s})^2|0\rangle = 0$ no identical 2-electron state (exclusion princ.)

a real wfn

$$|\Psi_2\rangle = \int d^3p\, d^3p'\, \Phi(p,p')\, \underbrace{b^\dagger_{p\uparrow} b^\dagger_{p'\uparrow}|0\rangle}_{\substack{|p\uparrow,\, p'\uparrow) \\ 2e^-\ \text{state}}} = -\int d^3p\, d^3p'\, \Phi(p,p')\, \underbrace{b^\dagger_{p'\uparrow} b^\dagger_{p\uparrow}|0\rangle}_{|p'\uparrow\, p\uparrow)}$$

$\therefore\ \Phi(p,p') = -\Phi(p',p)$ $\boxed{\therefore \text{F states are antisymmetric!}}$