Review for QM II Midterm

as usual: 3 problems, do all.
open books + notes.
bring a calculator

basic topics

scat: 1D barrier penetration.

3D potential scattering.

Born approx.

$\frac{d\sigma}{d\Omega}$, σ

partial waves and phase shifts $\{\delta_\ell\}$

Argand plots, Breit-Wigner resonances, def$^{\underline{n}}$ of resonance.

optical theorem.

S-matrix.

TDSE: eigenmode solution $\{a_m(t)\}$

Fermi's Golden Rule

radiative transitions

Specifics (what you should know)

1D barriers

procedure for solving for refl. & trans. amps r, t
(and internal wfns.) (Incl. δ-fn. potentials.)

cons. of prob. $|r|^2 + |t|^2 = 1$.

relations between r & t implied by $\mathcal{S}$-matrix.

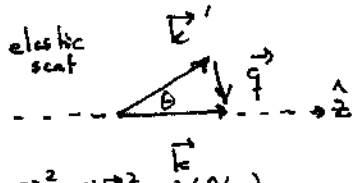
3D potential scat.

exact $\int$ eqn for scat. (Lippman - Schwinger eqn.)

$$\underbrace{\psi(\vec{x})}_{\substack{\text{exact}\\ \text{wfn.}}} = \underbrace{\psi^{(0)}(\vec{x})}_{\substack{\text{soln of}\\ \text{free Schrö}\\ \text{(plane wave)}}} + \int d^3x' \, \underbrace{K(\vec{x-x'})}_{\substack{\text{kernel,}\\ \text{free Schrö}}} \, V(\vec{x}') \, \underbrace{\psi(\vec{x}')}_{\substack{\text{exact}\\ \text{wfn.}\\ (=\text{integral form of}\\ \text{Schrö eqn.})}}$$

kernel, free Schrö eqn.

$$K(\vec{x-x'}) = -\frac{m}{2\pi\hbar^2} \frac{e^{ik|\vec{x-x'}|}}{|\vec{x-x'}|}$$



$\vec{q}^{\,2} = 4\vec{k}^2 \sin^2(\theta/2)$

Born series: far-field approx for wfn

$$\underbrace{\psi^{(1)}(\vec{x}) = -\frac{m}{2\pi\hbar^2} \underbrace{\int d^3x' \, V(\vec{x}') \, e^{i\vec{q}\cdot(\vec{x-x'})}}_{\text{F.T. } \tilde{V}(\vec{q})} \cdot \frac{e^{ikr}}{r}}_{f(\theta) \text{ only for radial } V(r) \text{ only}}$$

general results

$$\psi = \underbrace{e^{ikz}}_{\psi_{inc}} + \underbrace{f\frac{e^{ikr}}{r}}_{\psi_{scat}} \rightarrow \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

$$\sigma = \int |f(\Omega)|^2 d\Omega$$

<u>special cases Born approx</u>

$\vec{p} = \hbar \vec{k}$

Coulomb scat $\quad \dfrac{d\sigma}{d\Omega} = \dfrac{m^2 e^4}{4 p^{2}} \dfrac{1}{\sin^4(\theta/2)}$

<u>Rutherford</u> cross section. $\quad \sigma = \infty$.

screened Coulomb.

δ-function scat.

lattice scat + Bragg condition (prob set 3)

<u>PWA</u> $\quad \psi = \underbrace{e^{ikz}}_{\psi_{inc}} + \underbrace{f(\theta)\dfrac{e^{ikr}}{r}}_{\psi_{scat}}$ if $V(r)$ only

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\underbrace{e^{i\delta_\ell}\sin(\delta_\ell)}_{\equiv a_\ell} P_\ell(\cos\theta)$$

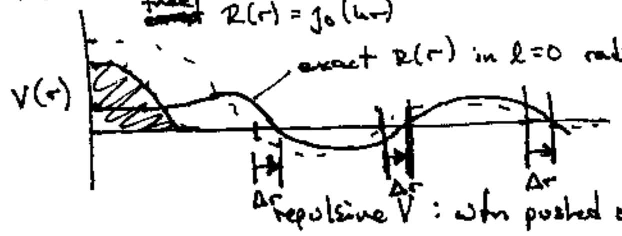
$\delta_\ell \equiv \ell^{th}$ partial wave phase shift.

$$\underline{\sigma} = \int |f(\theta)|^2 d\Omega = \underline{\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell}$$

$$\equiv \sum_{\ell=0}^{\infty}\sigma_\ell$$

$\therefore \sigma_\ell$ has the "<u>unitarity bound</u>" $\quad \underline{\sigma_\ell \leq \frac{4\pi}{k^2}(2\ell+1)}$
[reached at $\delta_\ell = 90°$]
very useful at low energies $\rightarrow$ S-wave scat $\rightarrow \sigma_0 \leq \frac{4\pi}{k^2}$

<u>determining phase shifts</u> (e.g. δ_0)



$V>0 \rightarrow \delta_\ell < 0$

$V<0 \rightarrow \delta_\ell > 0$ (if V weak)

repulsive V : wfn pushed out by Δr. $\quad \delta_0 = -k\Delta r$

true $\forall \ell$.

from inspection of exact external soln,

$$R(r)\Big|_{V(r)\ \text{negligible}} \simeq \underbrace{A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr)}$$

$$\hookrightarrow \tan\delta_\ell = -\frac{B_\ell}{A_\ell}$$

exact solns, hard sphere phase shifts.

$$\delta_0 = -kR \equiv -x$$

$$\delta_1 = -\tan^{-1}\left\{\frac{\tan(x)-x}{1+x\tan(x)}\right\}$$

$$\ldots \qquad \delta_\ell = \tan^{-1}\left\{\frac{j_\ell(x)}{\eta_\ell(x)}\right\}$$

threshold behavior

$$\left\{\begin{array}{ll} \delta_\ell \propto k^{2\ell+1} & \text{small } k \\ \therefore \sigma_\ell \propto k^{4\ell} & \text{small } k \end{array}\right.$$

generally

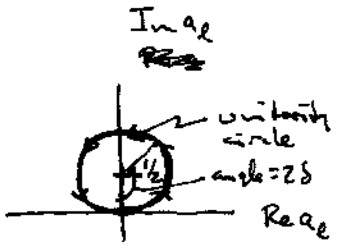
$$\lim_{k\to 0} \delta_0 = -ka$$

"scattering length"

Argand diagrams

$$f(\Omega) = \frac{1}{k}\sum_\ell (2\ell+1)\, a_\ell\, P_\ell(\mu)$$

$$\hookrightarrow \underbrace{\frac{1}{2i}\left(e^{2i\delta_\ell}-1\right)}_{\text{circle of radius } 1/2}$$



trajectory of a perfectly elastic resonance

$$\rightarrow \frac{1}{2i}\left(\eta_\ell e^{2i\delta_\ell}-1\right)$$

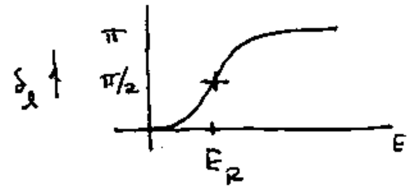
"inelasticity"

$$0 \le \eta_\ell \le 1$$

to allow for absorption

· Breit-Wigner resonance (E_R, Γ_R fixed)

$$a_\ell \propto \frac{1}{E_R - E - i\Gamma_R/2}$$



"resonant phase motion"

$$\left.\frac{d\delta_\ell}{dE}\right|_{E=E_R} = \frac{1}{\Gamma} = T =$$ "lifetime of the resonance"

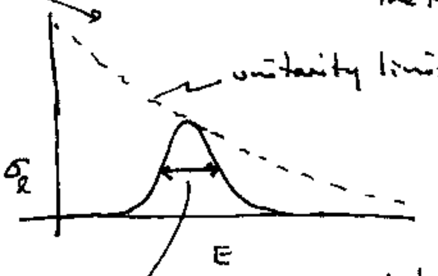
resulting σ

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\underbrace{\frac{\Gamma_R^2/4}{(E-E_R)^2+\Gamma_R^2/4}}_{\text{BW envelope}}$$

↑ note also has E dep.!

$k^2 = \frac{p^2}{\hbar^2} = \frac{2m}{\hbar^2}E$

$\frac{p^2}{2m} = E$



careful: FWHM of $E \cdot \sigma_\ell$ is Γ_R given a BW shape.

(usually σ is used, OK if $\Gamma_R \ll E_R$)

optical theorem

$$\sigma_{tot}(k) = \frac{4\pi}{k}\,\mathrm{Im}\,f_k(\theta=0)$$

(comes from conservation of flux: what's taken out of the forward beam is scattered)

<u>det. of phase shifts</u> (cont.)

given $V(r)$, <u>exact</u> radial wfn is $\quad R_{k,l}(r) \underset{r\to\infty}{\approx} \cos\delta_l\, j_l(kr) - \sin\delta_l\, \eta_l(kr)$

subst. into Lippmann-Schwinger eqn gives
the <u>exact</u> $\int$ eqn for phase shifts,

$$\sin\delta_l = -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, R_{k,l}(r)\, j_l(kr)\, r^2\, dr$$

<u>1st Born approx sets</u> $R_{k,l}(r) = j_l(kr)$ and assumes $\delta_l \ll 1$:

$$\delta_l^{(\text{Born})} = -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, j_l(kr)^2\, r^2\, dr$$

<u>S-waves</u>

$$\delta_0^{(\text{Born})} = -\frac{2m}{\hbar^2 k}\int_0^\infty V(r)\sin^2(kr)\, dr$$

<u>low energy limit,</u> Born approx

$$\lim_{k\to 0}\delta_l^{(\text{Born})}(k) = -\frac{2m}{\hbar^2}\left[\frac{2^l\, l!}{(2l+1)!}\right]^2 \underbrace{\int_0^\infty r^{2l+2}\, V(r)\, dr}_{\text{a radial moment of the potential}} \cdot \underbrace{k^{2l+1}}_{\text{expected } k^{2l+1} \text{ threshold behavior}}$$

<u>", S-waves</u>

$$\lim_{k\to 0}\delta_0^{(\text{Born})}(k) = -\frac{mk}{2\pi\hbar^2}\int d^3x\; V(r)\,.$$

$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0$ $\to \int d\Omega\, |f(\Omega)|^2$ ✓

prev Born cross sec, FT $V(r)$.

Optical thm.

Exact cons. of probability result, from comparing

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell$$

and

$$f(\Omega) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin\delta_\ell \, e^{i\delta_\ell} P_\ell(\cos\theta)$$

$$\underline{\sigma_{tot} = \frac{4\pi}{k} \, \mathrm{Im} \, f(0)}$$

S-matrix (Heisenberg 1942)

A formalism for expressing scat. amplitudes, contains all of physics in 1 matrix.

$$S_{fi} = \langle f | \psi(t=+\infty) \rangle = \langle f | i \rangle$$

└ specified initial state at $t = -\infty$

└ specified final state at $t = +\infty$

S = $n \times n$ unitary matrix for an n-state system. entries are $i \to f$ scat amps.

(implicit time evolution operator) $= \langle f | \underbrace{U(\infty, -\infty)}_{\text{time evol. op from } t=-\infty \text{ to } t=+\infty.} | i \rangle$

$S^\dagger = S^{-1}$ to conserve probability. $S = I + i \underbrace{T}_{\text{"T-matrix"}}$

$$\langle f | f \rangle = \langle i | S^\dagger S | i \rangle = \langle i | i \rangle \text{ if } S^\dagger = S^{-1}$$

1D barrier probs

$$S = \begin{bmatrix} t & r \\ r & t \end{bmatrix}$$

$S^\dagger S = I$ implies r & t relatively <u>Imaginary</u>.

if diagonal,

$$\mathcal{S} = \begin{bmatrix} e^{i\delta_0} & & & \\ & e^{i\delta_1} & & \\ & & \ddots & \\ & & & e^{i\delta_n} \end{bmatrix}$$

is pure phases. (e.g. the phase shifts in our potential scat. problem.

Generally it will consist of square blocks for states with a common value of a conserved quantum number.

TDSE

$$i\hbar\dot{\psi}(\vec{x},t) = H\psi(\vec{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right]\psi(\vec{x},t)$$

formal soln

$$\psi(\vec{x},t) = \underbrace{e^{-\frac{i}{\hbar}H_{op}t}}_{U(t,0)}\psi(\vec{x},0)$$

"time translation operator" $(U^\dagger = U^{-1})$

$$U(\delta t,0) \approx 1 - \frac{i}{\hbar}\delta t \cdot \underbrace{H_{op}}_{\text{generator of time translations}}$$

Approaches to soln: (& choice of picture)

1. solve $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$ directly

(matrix DE, $n\times n$ H, for n-state system)

$$\langle \mathcal{O}\rangle(t) = \langle\psi(t)|\mathcal{O}|\psi(t)\rangle$$

or since this is

$$\langle\psi(0)|\underbrace{e^{+iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar}}|\psi(0)\rangle$$

You can call this $\mathcal{O}(t)$ & leave the states fixed.
≡ Heisenberg picture.

(Usual $\mathcal{O}$ fixed, $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$ is the "Schrödinger picture.)

To evaluate
$$\mathcal{O}(t) = e^{iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar}$$

use the nested commutator identity
$$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots$$

equivalent differential form
$$-i\hbar\dot{\mathcal{O}}(t) = [H,\mathcal{O}(t)]$$

<u>time dep.</u>
<u>pert thy.</u>

$$\underline{S_{fi}} = \langle f|\psi(t=+\infty)\rangle = \langle f|U(\infty,-\infty)|i\rangle$$

$$\text{If} \quad H = H_0 + \underbrace{H_I(t)}_{\text{small \& time dep.}}$$

you can derive a series expansion for the eigenmode coefficients $\{a_n(t)\}$ in
$$|\psi(t)\rangle = \sum_n a_n(t)\,\underbrace{|\psi_n(t)\rangle}_{\underbrace{|\psi_n(0)\rangle e^{-iE_n^{(0)}t/\hbar}}_{H_0 \text{ eigenstates}}} \qquad \omega_n \equiv E_n^{(0)}/\hbar$$

which is
$$i\hbar\dot{a}_m(t) = \sum_n \langle m|H_I(t)|n\rangle e^{i(\omega_m-\omega_n)t}\,a_n(t)$$
(matrix DE)

$a_m(t) = a_m^{(0)} + a_m^{(1)}(t) + \dots$

leading amplitude is
$$a_m^{(1)}(t) = \underbrace{\delta_{mi}}_{a_m^{(0)}(t)} - \frac{i}{\hbar}\int_{-\infty}^{t} \langle m|H_I(t')|i\rangle e^{i(\omega_m-\omega_i)t'}\,dt' + \mathcal{O}(H_I^2)$$

$a_f(+\infty)$ gives the $i\to f$ transition amp. $\equiv S_{fi}$

Explicit $H_I(t)$ time dependence is required for a $\Delta E_{fi} \neq 0$ transition.

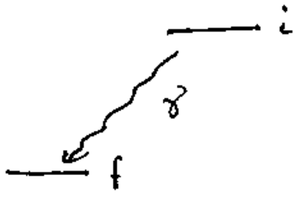
(we did some time-dep. pulse examples).

Fermi's
Golden Rule

from $a_{fi} \to P_{fi} = |a_{fi}|^2$ we derived the transition rate into a continuum,

$$d\Gamma_{i\to f} = \frac{2\pi}{\hbar} |\langle f | H_I | i \rangle|^2 \, \underbrace{d n}_{\substack{\text{differential number of states in} \\ \text{element of final phase space,} \\ \text{e.g. } V \frac{d^3k}{(2\pi)^3} \\ \text{for photons of 1 polarization.}}} \qquad \delta(E_f - E_i)$$

Radiative transitions



$$H_I = -\frac{1}{c} \int d^3x \, \vec{j} \cdot \vec{A}$$

we derived the transition rate between initial and final atomic states, starting from the golden rule.

also used nice tricks like $[H, \vec{x}] = -\frac{i\hbar}{m} \vec{p}$ to simplify the result.

result

$\vec{A}$ plane wave ≈ 1

$$\frac{d\Gamma^{E1}}{d\Omega} = \frac{1}{2\pi} \alpha c \, |\hat{\epsilon}^*_{k\lambda} \cdot \vec{x}_{fi}|^2 \, k_\gamma^3$$

doing the polarization sum $\sum_\lambda |\hat{\epsilon}^*_{k\lambda} \cdot \vec{v}|^2 = |\vec{v}|^2 - |\hat{k} \cdot \vec{v}|^2$ + integrating over angles gave

$$\boxed{\Gamma^{E1}_{A\to B\gamma} = \frac{4}{3} \alpha c \, |\vec{x}_{fi}|^2 \, k_\gamma^3}$$

A hydrogen e.g. $\Gamma_{2P\to 1S+\gamma} = \left(\frac{2}{3}\right)^8 \frac{mc^2}{\hbar} \alpha^5$