Scattering amplitudes and Argand diagrams

The partial wave decomposition represents the scat. amplitude $f(\hat{\Omega})$ as the expansion

$$f(\hat{\Omega}) = \frac{1}{k}\sum_{\ell}(2\ell+1)\, a_\ell\, P_\ell(\mu)$$

$$a_\ell = \underbrace{\frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)}_{\substack{\uparrow \qquad\quad \uparrow \\ \text{full wfn} \;-\; \text{inc. plane wave}}}$$

This amplitude a_ℓ is often shown as a trajectory (versus energy) in complex-a_ℓ space, called an Argand plot (or Argand diagram).

a_ℓ is generalized somewhat to the case in which some of the full wfn strength $e^{2i\delta_\ell}$ has been absorbed in the scatterer. (apparent violation of conservation of probability, because we are not listing all possible states:

$$a_\ell = \frac{1}{2i}\left(\eta_\ell\, e^{2i\delta_\ell} - 1\right)$$

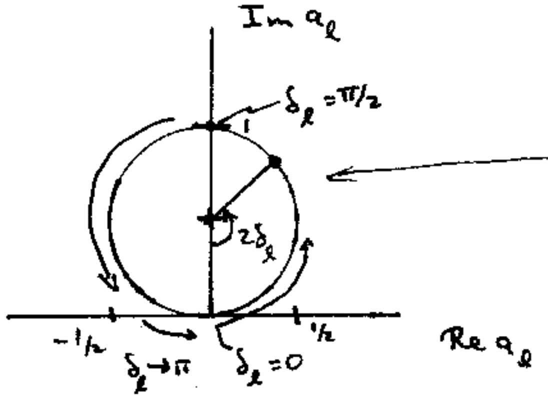
$\uparrow$ real "inelasticity", $0 \le \eta_\ell \le 1$

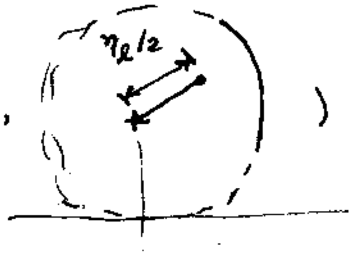
(pure elastic scat is $\eta_\ell = 1$)

Note for elastic scat,

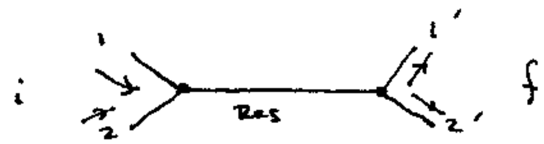
$$a_\ell = \frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)$$

this is a point on a circle of radius $1/2$, called the "unitarity circle". For $\delta_\ell = 0$ (typical threshold) $a_\ell = 0$, as δ_ℓ runs through $0 \to \pi/2$, it (a_ℓ) moves counterclockwise along the circle to the top.
As $\delta_\ell = \frac{\pi}{2} \to \pi$ (completing a "resonance loop") it comes back to the origin.



(if inelastic, )

This is exactly the behavior we would expect if the scattering were due to an intermediate resonance.
(e.g. collision of 2 particles):



the scat. amps can be calculated in pert. theory, which leads to sums over intermediate states times energy denominators:

$$a_\ell \sim \sum_m \frac{\langle f|H_I|m\rangle\langle m|H_I|i\rangle}{E - E_m} \to \frac{\langle f|H_I|Res\rangle\langle Res|H_I|i\rangle}{E - E_{Res}}$$

if dominated by a single Res state.

Note a resonance is not an energy eigenstate $\propto e^{-iE_R t/\hbar}$, since it decays.

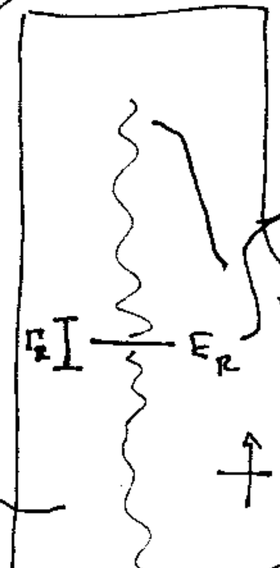
$$P_{Res} \propto e^{-\Gamma_R t/\hbar}, \text{ so } \quad \psi_{res} \propto e^{\underbrace{-i(E_R - i\Gamma_R/2)t}}$$

which suggests that a resonance gives an energy denominator with a pole of the real axis,

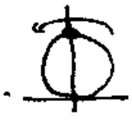
$$a_\ell^{(Res)} \propto \frac{1}{E_R - E - i\Gamma_R/2}$$

consider the limits of this $\delta a_\ell^{(Res)}$: $|E - E_R|$ large, $\gg \Gamma/2$, but $E \ll E_R$.

$a_\ell \rightarrow$ real, small, positive

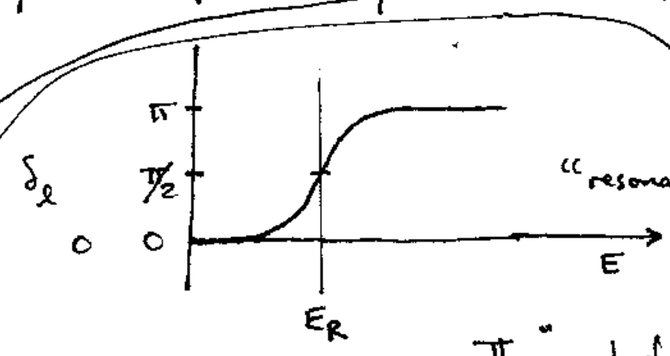


$E \rightarrow E_R$ from below, $a_\ell \rightarrow \frac{1}{-i\Gamma_R/2}$ pure Im.



Phase has rotated to pure Im.

$E \gg E_R$, again real, comes back to 0 from opposite side.

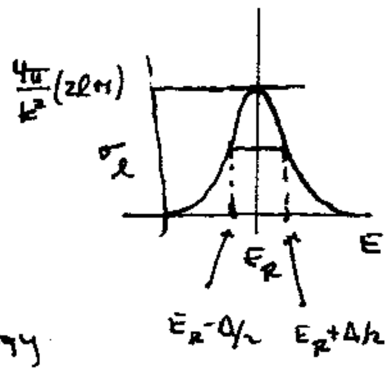


"resonant phase motion"

The "speed of passage" through the resonance energy, $\left.\frac{d\delta_\ell}{dE}\right|_{E_R} = \frac{1}{\Gamma} = T =$ "lifetime of the resonance"

This "Breit-Wigner" resonance amplitude also leads to a characteristic form for the cross section,

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\cdot\frac{\Gamma^2/4}{(E-E_R)^2+\Gamma^2/4}$$



which follows from unitarity and $|a_l|^2$ of the energy denominator.

note the "FWHM" Δ (full width half max) is given by

$$\frac{\Gamma^2/4}{\left(\frac{\Delta}{2}\right)^2+\frac{\Gamma^2}{4}} \equiv \frac{1}{2}, \quad \therefore \quad \underline{\Delta = \Gamma}.$$

This is often used as an estimate of the total width Γ, hence the lifetime $\tau = \hbar/\Gamma$, of a resonance.

The second form is obtained using the identity $dy/dp_z = 1/E$, and the third form represents the average over ϕ.

Feynman's x variable is given by

$$x = \frac{p_z}{p_{z\,\max}} \approx \frac{E+p_z}{(E+p_z)_{\max}} \quad (p_T \ll |p_z|) \; . \tag{37.39}$$

In the c.m. frame,

$$x \approx \frac{2p_{z\,\rm cm}}{\sqrt{s}} = \frac{2m_T \sinh y_{\rm cm}}{\sqrt{s}} \tag{37.40}$$

and

$$= (y_{\rm cm})_{\max} = \ln(\sqrt{s}/m) \; . \tag{37.41}$$

For $p \gg m$, the rapidity [Eq. (37.37)] may be expanded to obtain

$$\begin{aligned} y &= \frac{1}{2}\ln\frac{\cos^2(\theta/2) + m^2/4p^2 + \ldots}{\sin^2(\theta/2) + m^2/4p^2 + \ldots} \\ &\approx -\ln\,\tan(\theta/2) \equiv \eta \end{aligned} \tag{37.42}$$

where $\cos\theta = p_z/p$. The pseudorapidity η defined by the second line is approximately equal to the rapidity y for $p \gg m$ and $\theta \gg 1/\gamma$, and in any case can be measured when the mass and momentum of the particle is unknown. From the definition one can obtain the identities

$$\sinh\eta = \cot\theta \;\; , \quad \cosh\eta = 1/\sin\theta \;\; , \quad \tanh\eta = \cos\theta \; . \tag{37.43}$$

37.5.3. ***Partial waves***: The amplitude in the center of mass for elastic scattering of spinless particles may be expanded in Legendre polynomials

$$f(k,\theta) = \frac{1}{k}\sum_{\ell}(2\ell+1)a_\ell P_\ell(\cos\theta) \; , \tag{37.44}$$

where k is the c.m. momentum, θ is the c.m. scattering angle, $a_\ell = (\eta_\ell e^{2i\delta_\ell} - 1)/2i$, $0 \le \eta_\ell \le 1$, and δ_ℓ is the phase shift of the ℓ^{th} partial wave. For purely elastic scattering, $\eta_\ell = 1$. The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \; . \tag{37.45}$$

The optical theorem states that

$$\sigma_{\rm tot} = \frac{4\pi}{k}{\rm Im}\, f(k,0) \; , \tag{37.46}$$

and the cross section in the ℓ^{th} partial wave is therefore bounded:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)|a_\ell|^2 \le \frac{4\pi(2\ell+1)}{k^2} \; . \tag{37.47}$$

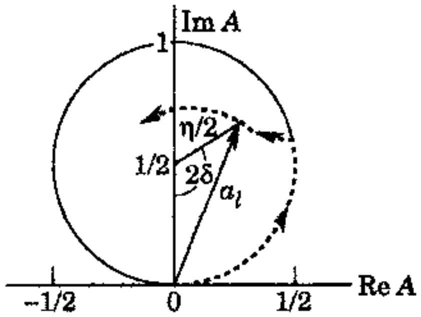


Figure 37.6: Argand plot showing a partial-wave amplitude a_ℓ as a function of energy. The amplitude leaves the unitary circle where inelasticity sets in ($\eta_\ell < 1$).

The evolution with energy of a partial-wave amplitude a_ℓ can be displayed as a trajectory in an Argand plot, as shown in Fig. 37.6.

The usual Lorentz-invariant matrix element $\mathscr{M}$ (see Sec. 37.3 above) for the elastic process is related to $f(k,\theta)$ by

$$\mathscr{M} = -8\pi\sqrt{s}\, f(k,\theta)\ , \tag{37.48}$$

so

$$\sigma_{\rm tot} = -\frac{1}{2p_{\rm lab}\, m_2}\ {\rm Im}\,\mathscr{M}(t=0)\ , \tag{37.49}$$

where s and t are the center-of-mass energy squared and momentum transfer squared, respectively (see Sec. 37.4.1).

37.5.3.1. *Resonances:* The Breit-Wigner (nonrelativistic) form for an elastic amplitude a_ℓ with a resonance at c.m. energy E_R, elastic width $\Gamma_{\rm el}$, and total width $\Gamma_{\rm tot}$ is

$$a_\ell = \frac{\Gamma_{\rm el}/2}{E_R - E - i\Gamma_{\rm tot}/2}\ , \tag{37.50}$$

where E is the c.m. energy. As shown in Fig. 37.7, in the absence of background the elastic amplitude traces a counterclockwise circle with center $ix_{\rm el}/2$ and radius $x_{\rm el}/2$, where the elasticity $x_{\rm el} = \Gamma_{\rm el}/\Gamma_{\rm tot}$. The amplitude has a pole at $E = E_R - i\Gamma_{\rm tot}/2$.

The spin-averaged Breit-Wigner cross section for a spin-J resonance produced in the collision of particles of spin S_1 and S_2 is

$$\sigma_{BW}(E) = \frac{(2J+1)}{(2S_1+1)(2S_2+1)}\ \frac{\pi}{k^2}\ \frac{B_{\rm in}B_{\rm out}\Gamma_{\rm tot}^2}{(E-E_R)^2+\Gamma_{\rm tot}^2/4}\ , \tag{37.51}$$

where k is the c.m. momentum, E is the c.m. energy, and $B_{\rm in}$ and $B_{\rm out}$ are the branching fractions of the resonance into the entrance and exit channels. The $2S+1$ factors are the multiplicities of the incident spin states, and are replaced by 2 for photons.

Optical theorem

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell$$

note also

$$f(\Omega) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\cos\theta)$$

so the <u>forward</u> scattering amplitude, $\theta = 0$ satisfies

$$f(\theta=0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, \underbrace{P_\ell(1)}_{1}$$

so $$\mathrm{Im}\, f(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell$$

and hence $$\boxed{\sigma_{tot} = \frac{4\pi}{k}\, \mathrm{Im}\, f(0)}$$

This very fundamental result is called the "optical theorem".

A priori this looks very funny; it relates a cross section ($|\text{amplitude}|^2$) to an amplitude (Im part) in a particular direction (forward, $\theta=0$).

Actually this is a statement of conservation of probability:
Handwaving justification

$$\underbrace{P_{scat}}_{} = 1 - P_{not\ scat} \longrightarrow P_{not\ scat} \propto |\psi(\theta=0)|^2$$

$$= \left| e^{ikz} + f(0)\frac{e^{ikr}}{r} \right|^2$$

$$\propto \sigma_{tot} \propto 1 - \left(1 - c_1 \overset{\mathrm{Im}}{f(0)}\right)$$

$$= 1 + c_1 \underline{\frac{f(0)}{r}} + \underline{c_2 \frac{|f(0)|^2}{r^2}}$$

actually involves Im, $\frac{|f(0)|^2}{r^2}$ doesn't contribute due to $r \to \infty$.

$$\propto \mathrm{Im}\, f(0)$$

+ actual normalization is as above.