<u>Intuitive picture of phase shifts</u> ($\delta_\ell < 0$ for $\underset{\text{weak}}{\wedge}$ $V > 0$ typically, e.g.).

For a free partical the radial wfn. of angular momentum ℓ is

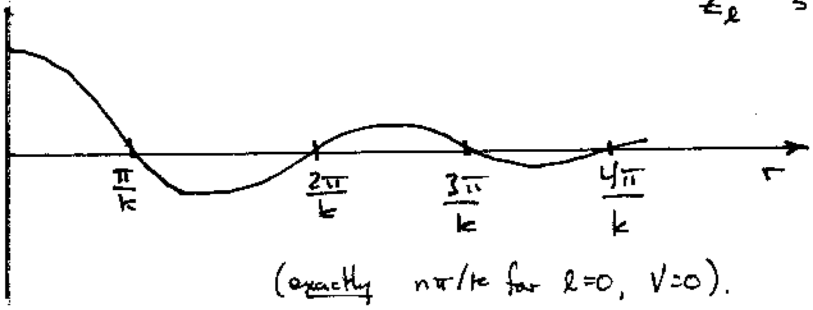
$$R_{\ell,k}(r) \longrightarrow j_\ell(kr) \underset{r\to\infty}{\approx} \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right)$$

and successive zeros are at $\quad k z_\ell^{(n)} - \frac{\ell\pi}{2} = n\pi$

$$z_\ell^{(n)} = \frac{(n+\frac{1}{2}\ell)\pi}{k}$$

$R_{\ell,k}(r)$

$\ell = 0 \to j_0$



(exactly $n\pi/k$ for $\ell=0$, $V=0$).

$\ell=1 \to j_1$

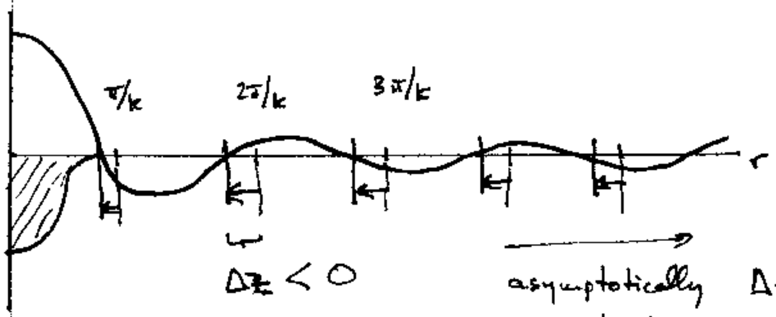
(asymp. at $\frac{(n+\frac{1}{2})\pi}{k}$)

etc

now suppose we turn on an attractive potential.
The wfn will be "pulled in" towards smaller r;

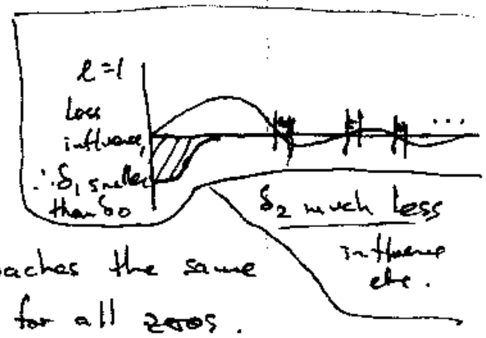
$R_{\ell,k}(r)$

$V(r)$



$\Delta z < 0$

asymptotically Δz approaches the same constant as $r \to \infty$, for all zeros.



our def$^{\underline{n}}$ for phase shifts was $\underline{\lim_{r\to\infty} R_{\ell,k}(r) \approx \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2} + \underline{\underline{\delta_\ell}}\right)}$

$\therefore$ zeros are at $\quad k z_\ell^{(n)}(V) - \frac{\ell\pi}{2} + \delta_\ell = n\pi$

$$z_\ell^{(n)}(V) = \underbrace{\frac{(n+\frac{1}{2}\ell)\pi}{k}}_{z_\ell^{(n)}(V=0)} - \underbrace{\frac{\delta_\ell}{k}}_{\text{shift in zero position due to potential}}$$

$$\therefore\ \underset{\sim}{\delta_\ell} = -k\left[z_\ell^{(n)}(V) - z_\ell^{(n)}(0)\right] = -k\,\Delta z_\ell^{(n)} = \underset{\sim}{-k\,\Delta z_\ell}$$

(indep of n as $n \to \infty$)

So, the def^n implies

$\Delta z_\ell < 0$ (ψ pulled in, typical of attractive V)

$\to \delta_\ell > 0$ (pos. phase shift)

$\Delta z_\ell > 0$ (ψ pushed out, typ. of repulsive V)

$\to \delta_\ell < 0$ (neg. phase shift)

with the caveat that large p.s. of potentials can be misleading due to quadrant "wraparound"

Hard sphere phase shifts

Obviously the Born series in V fails completely if $V \to \infty$ over some finite region. Thus the "hard sphere" potential

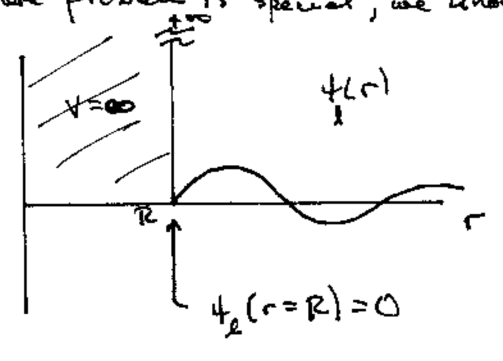
$$V(r) = \begin{cases} \infty & r < R \\ 0 & r > R \end{cases}$$

is an excellent problem to illustrate the usefulness of the partial wave expansion – Born can't say anything here.

Recall to get the phase shifts $\{\delta_\ell\}$ we solve the full radial problem outside the potential,

$$\lim_{r\to\infty} \psi_\ell(r) = \underbrace{A_\ell\, j_\ell(kr) + B_\ell\, \eta_\ell(kr)}_{\tan\delta_\ell = -\frac{B_\ell}{A_\ell}\ \text{from asymp. behavior}} \cong \underbrace{C_\ell}_{\substack{=1 \\ \text{defines} \\ \text{overall} \\ \text{norm}}} \frac{1}{kr} \sin\Big(kr - \frac{\ell\pi}{2} + \underbrace{\delta_\ell}_{\substack{\text{normally we use this} \\ \text{(zero crossings)}}}\Big)$$

hard sphere problem is special, we know A_ℓ & B_ℓ exactly:



$$\therefore -\frac{B_\ell}{A_\ell} = +\frac{j_\ell(kR)}{\eta_\ell(kR)} = \tan\delta_\ell$$

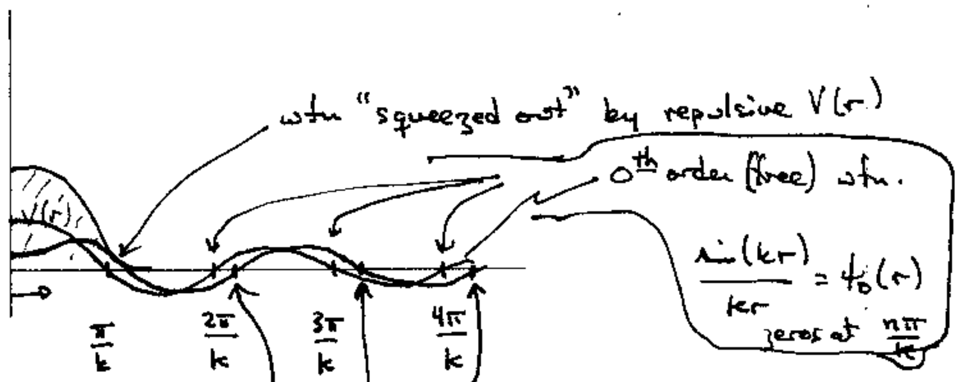
$$\delta_\ell = \tan^{-1}\left\{\frac{j_\ell(kR)}{\eta_\ell(kR)}\right\}$$

S-wave

$$\tan\delta_0 = \frac{j_0(kR)}{\eta_0(kR)} = \frac{\Delta/kR}{-e/kR} = -\tan kR$$

$$\underline{\delta_0 = -kR}$$

repulsive potential → negative phase shift. Why?



new zeros at $kr_n + \delta_0 = n\pi$

$$r_n = \underbrace{\frac{n\pi}{k}}_{r_n\ (V=0)} - \underbrace{\frac{\delta_0}{k}}_{\delta_0<0 \text{ if } r_n \text{ increases}}$$

[similarly, an attractive potential implies a positive phase shift]. (this works for small phase shifts. can be confusing when you cross quadrants)

P-wave

$$\tan\delta_1 = \frac{j_1(kR)}{\eta_1(kR)} = \frac{\frac{\Delta}{x^2}-\frac{e}{x}}{-\frac{e}{x^2}-\frac{\Delta}{x}} = -\frac{\frac{1}{x^2}T-\frac{1}{x}}{\frac{1}{x^2}+\frac{T}{x}} = -\left(\frac{T-x}{1+xT}\right)$$

$$\delta_1 = -\tan^{-1}\left\{\frac{\tan(kR)-kR}{1+kR\tan(kR)}\right\}$$

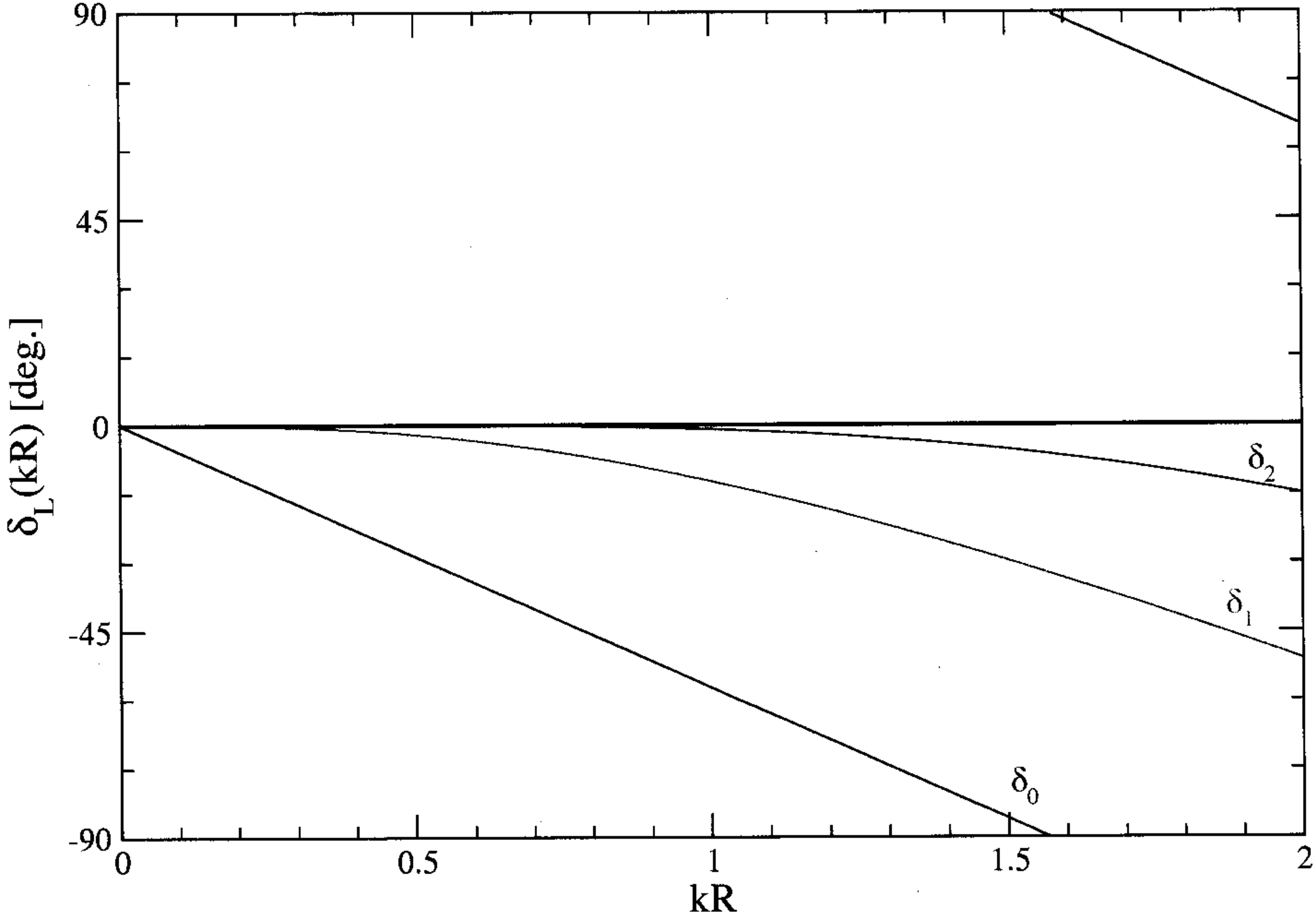
D-wave $$\tan\delta_2 = \frac{j_2}{\eta_2} = \frac{-\left[\left(\frac{3}{x^3}-\frac{1}{x}\right)\Delta - \frac{3}{x^2}e\right]}{\left[\left(\frac{3}{x^3}-\frac{1}{x}\right)e + \frac{3}{x^2}\Delta\right]} = -\frac{\left[T - \frac{3/x^2}{3/x^3-1/x}\right]}{\left[1+\frac{3/x^2}{3/x^3-1/x}T\right]}$$

$$\delta_2 = -\tan^{-1}\left\{\left[T-\left(\frac{x}{1-x^2/3}\right)\right] \Big/ \left[1+\left(\frac{x}{1-x^2/3}\right)T\right]\right\}$$

Hard sphere phase shifts.

$\delta_\ell = \tan^{-1}\left\{ \frac{j_\ell(kR)}{\eta_\ell(kR)} \right\}$

L=0,1,2



hard sphere
oops radius is called R ~~or~~ a
in my notes.

<u>General ℓ result</u>

$$\tan\delta_\ell = \frac{j_\ell(ka)}{\eta_\ell(ka)}$$

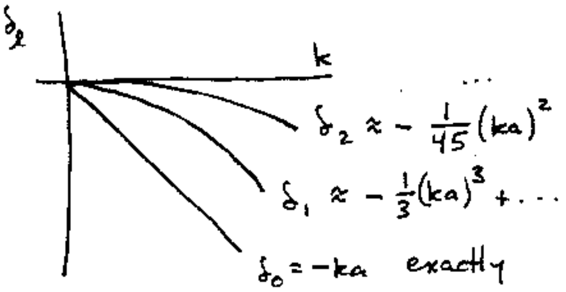
so, taking low energy (small k) approx

$$\lim_{ka\to 0} \quad \delta_\ell \approx -\frac{(ka)^{2\ell+1}}{(2\ell+1)\left[(2\ell-1)!!\right]^2}$$

note $\delta_\ell \propto k^{2\ell+1}$ threshold behavior

$\therefore \ \sigma_\ell \propto k^{4\ell}$ threshold behavior

hard sphere e.g.

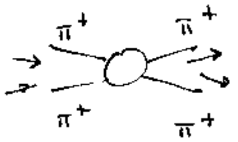


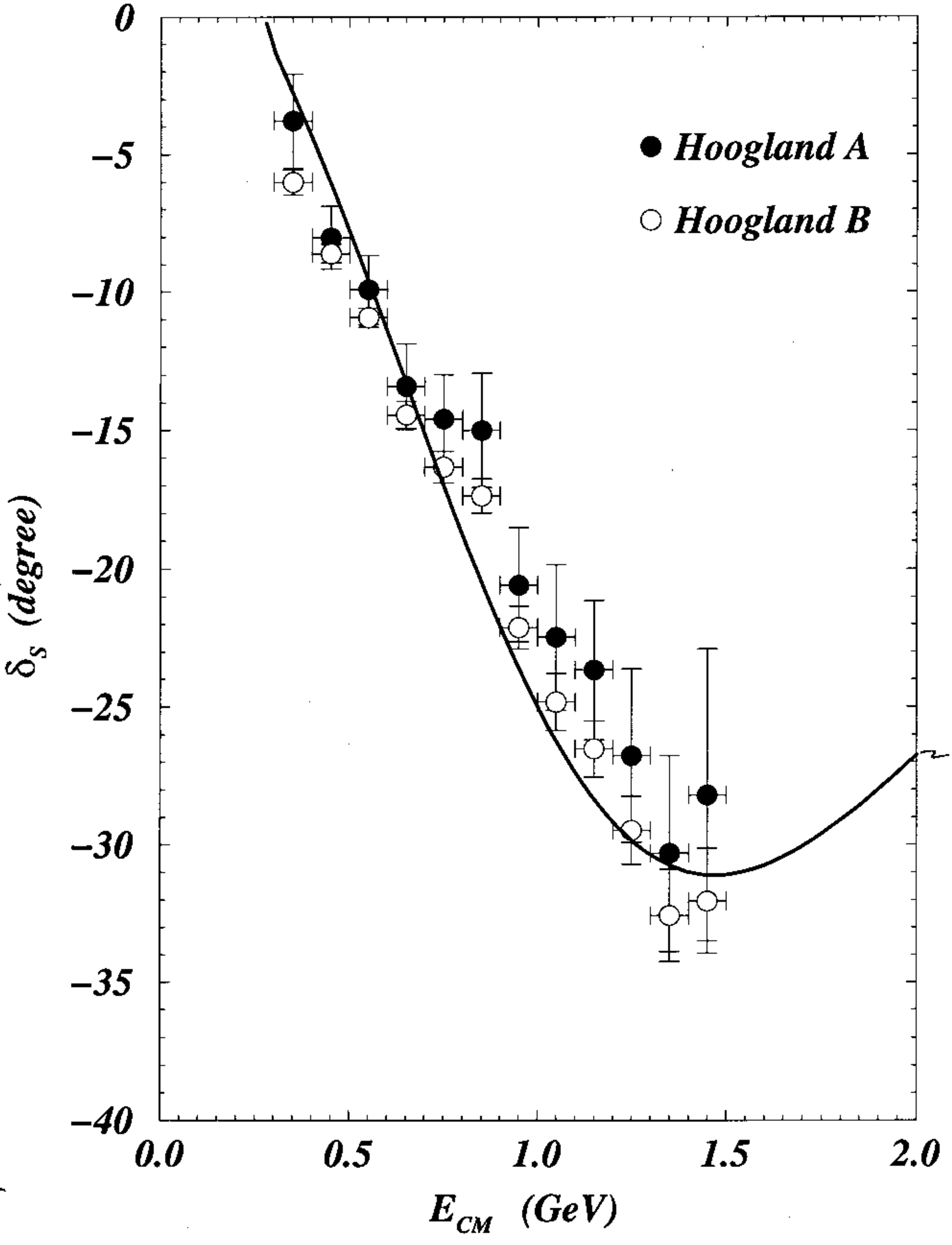
(more general than this example! typical of localized potentials.)

[asymp. behavior $\displaystyle\lim_{x\to 0}$ $j_\ell(x) \approx \dfrac{2^\ell \ell!}{(2\ell+1)!} x^\ell + \mathcal{O}(x^{\ell+2})$

$$\eta_\ell(x) \approx -\frac{(2\ell)!}{2^\ell \ell!} x^{-\ell-1} + \mathcal{O}(x^{-\ell+1}) \quad]$$

I=2 ππ S-wave phase shift
(e.g. $\pi^+\pi^+$)





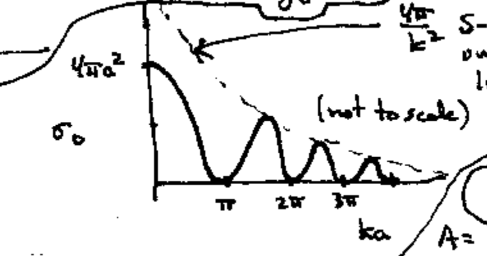
theory by yours truly et al. :)

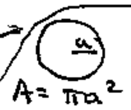
Potential radius is R $\underset{\equiv}{\text{or}}$ a. (Sorry I switched notation.)

<u>Cross section</u> (low energy limit) $\left[\sigma = \sum_{l=0}^{\infty} \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \text{ generally}\right]$

$ka \to 0$, $\therefore$ S-wave ($l=0$) only matters,

$$\sigma \approx \underbrace{\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2(ka)}_{\text{exactly}} = \boxed{4\pi a^2 j_0(ka)^2}$$





$$\lim_{k\to 0} \sigma = \frac{4\pi}{k^2}\cdot k^2 a^2 = 4\pi a^2$$

$$= 4 \times \text{apparent surface area.}$$

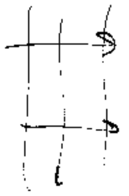
As k increases, expect higher partial waves to be important. It turns out that scattering is important for $l \lesssim ka$, $l_{cutoff} \sim ka$, which has the handwaving justification that at most

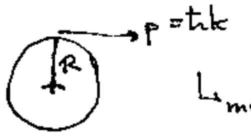
$$L_{orb} \sim \hbar l$$

can be made classically by incident particle of momentum $p = \hbar k$ scattering with impact parameter a,

$$\hbar l \sim \hbar k \cdot a$$

$$|\vec{L}| = |\vec{r} \times \vec{p}|.$$





$L_{max} \sim pR \sim \hbar k R$

$(R = a)$

The <u>exact</u> cross section for scattering from the ∞ square well can be written too (as a sum), since we know

$$\tan\delta_l = \frac{j_l(ka)}{\eta_l(ka)}.$$

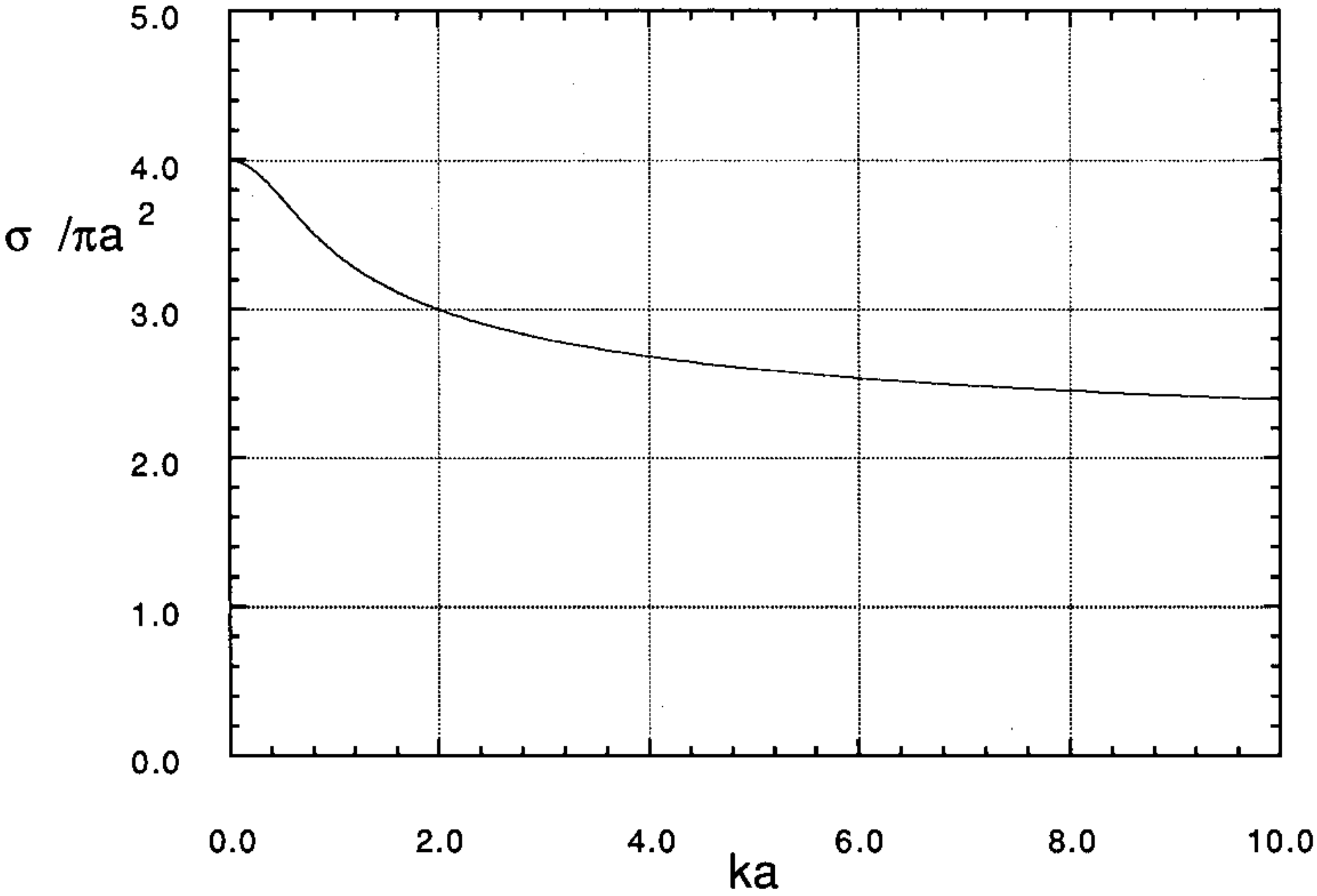
This exact result is

$$\boxed{\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\frac{j_l(ka)^2}{j_l(ka)^2 + \eta_l(ka)^2}}$$

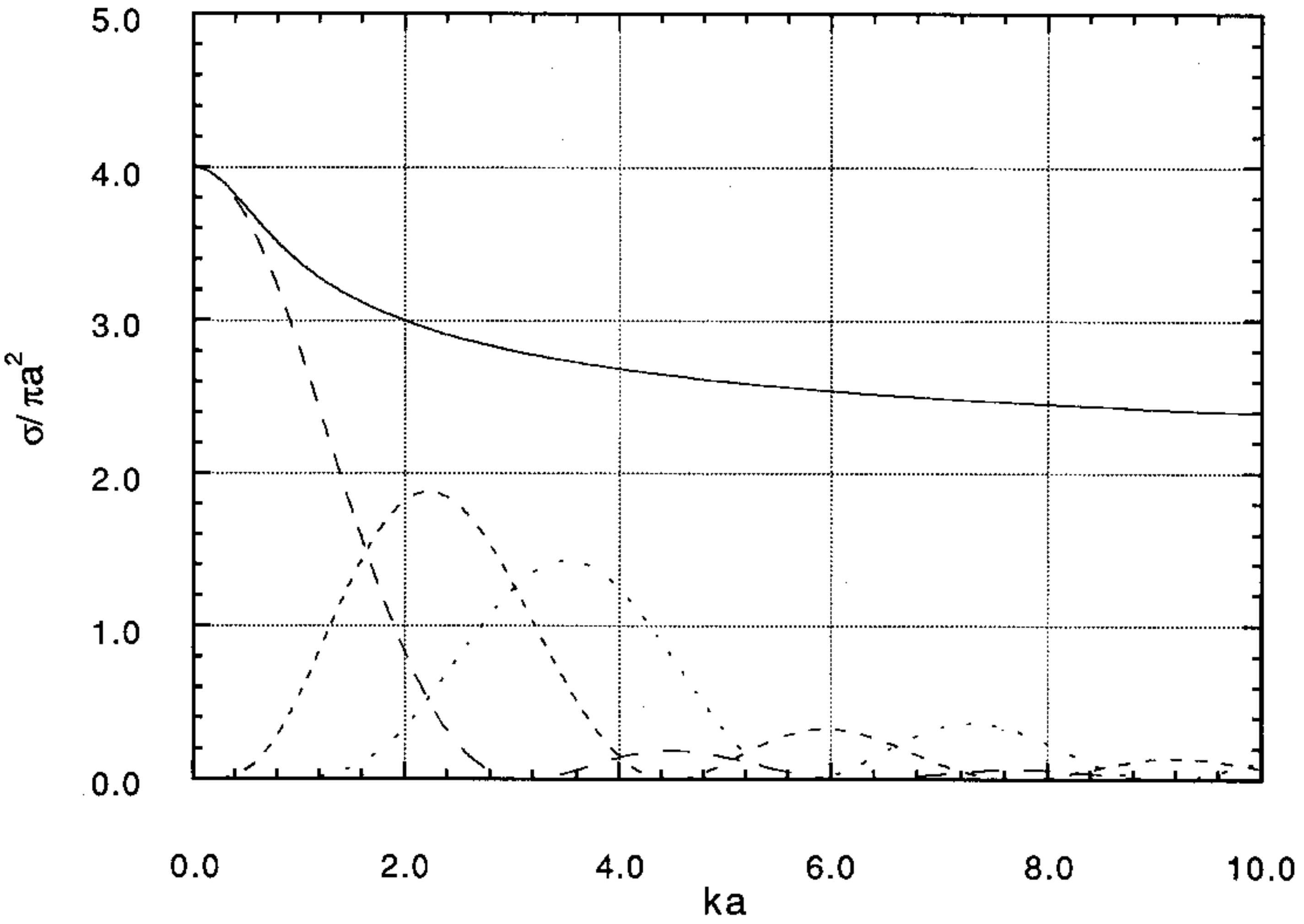
<u>Low E limit</u> $\boxed{\sigma_{tot} = 4\pi a^2}$ (any).

<u>High E limit</u> (much harder, all partial waves present) $\boxed{\sigma_{tot} = 2\pi a^2}$ Only >0 because of ∞ square well.

Hard sphere (radius=a) total cross section versus ka.



Hard sphere partial (L=0,1,2) and total cross sections.



Determination of phase shifts

How (other than taking $r \to \infty$ or fitting b.c.) do we actually determine phase shifts?

One approach is to return to our integral equation for $\psi(\vec{x})$ & extract them from there.

$$\psi_k(\vec{x}) = e^{ikz} + \underbrace{\int d\vec{x}'\, G_k(\vec{x}-\vec{x}')\, V(\vec{x}')\, \psi_k(\vec{x}')}_{f(\Omega)\frac{e^{ikr}}{r} \text{ as } r\to\infty} \qquad (1)$$

the "Lippmann-Schwinger" equation (exact $\int$ eqn for scattered wfn). $\left[e^{ikz} \to \text{"}\psi^{(0)}(\vec{x})\text{"}, \text{ soln of } V=0 \text{ schrö eqn} \right]$

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^{\ell} (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

$$\psi_k(\vec{x}) = \overbrace{\sum_{\ell=0}^{\infty} e^{i\delta_\ell} i^{\ell} (2\ell+1) \underbrace{R_{k,\ell}(r)}_{\text{asymptotically } = \underbrace{\{\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, \eta_\ell(kr)\}}_{\underline{\text{real}}\text{ radial wfn, asymp. form } r\to\infty}} P_\ell(\cos\theta)}^{\text{this is the "distorted plane wave" we found for the general wfn.}}$$

Substitute this expansion into r.h.s. of (1), with $R_{k,\ell}(r')$ and $V(\underline{\underline{r'}})$ only, then you find a result for $f(\Omega)$

$$f(\Omega) = -\frac{2m}{\hbar^2} \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} P_\ell(\cos\theta) \cdot \int_0^\infty V(r) R_{k,\ell}(r) j_\ell(kr) r^2 dr$$

So
$$\boxed{\sin\delta_\ell = -\frac{2mk}{\hbar^2} \int_0^\infty V(r) R_{k,\ell}(r) j_\ell(kr) r^2 dr}$$

<u>exactly</u>.

This is fine if we know the exact radial wfn, of course we don't in general. We can get approximate results by assuming e.g. that the potential is weak, so to lowest order

$$R_{k,\ell}(r) = j_\ell(kr) + \dots \qquad (V(r)\ \text{small})$$

and δ_ℓ is small too, so $\sin\delta_\ell \approx \delta_\ell$.

In which case

$$\underline{\delta_\ell \approx -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, j_\ell(kr)^2\, r^2\, dr} \qquad (2) \quad \underline{= \delta_\ell^{(\text{Born})}}$$

To understand this consider e.g. S-waves,

$$\delta_0 \approx -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\,\frac{\sin^2(kr)}{k^2}\,dr$$

assume also that the potential is very short range, so we can assume $kr \ll 1$ $\left(a_{\text{potential}} \ll \frac{1}{k} \sim \lambda_{\text{de Broglie}}\right)$.
Then

$$\delta_0 \approx -\frac{2mk}{\hbar^2}\int_0^\infty r^2 V(r)\,dr = -\frac{mk}{2\pi\hbar^2}\int V(r)\,d\vec{x}$$

and (assuming that higher partial waves are unimportant),

$$\sigma_{tot} \approx \sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{\cancel{4\pi}}{\cancel{k^2}}\cdot\frac{m^2\cancel{k^2}}{\cancel{4}\pi^2\hbar^4}\left|\int V(r)\,d\vec{x}\right|^2$$

$$= \frac{m^2}{\pi\hbar^4}\left|\int V(r)\,d\vec{x}\right|^2 = \frac{m^2}{\pi\hbar^4}\left|V(\vec{q}=\vec{0})\right|^2$$

Recall Born approx $\frac{d\sigma}{d\Omega} = \left|-\frac{m}{2\pi\hbar^2}V(\vec{q})\right|^2$, so $\sigma_0 = 4\pi\cdot\frac{m^2}{4\pi^2\hbar^4}\left|V(\vec{q}=0)\right|^2 = \text{above}$ ✓

Hence the δ_ℓ in (2) above is called the Born approximation for the phase shift, $\delta_\ell^{(\text{Born})}$.

The Born-order phase shifts

$$\delta_\ell^{(Born)} = -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, j_\ell(kr)^2 r^2\, dr$$

still requires integrals we typically cannot do analytically.

The extreme low-energy limit however IS tractible usually:

$$\lim_{k\to 0} j_\ell(kr) \approx \frac{2^\ell \ell!}{(2\ell+1)!}(kr)^\ell,$$

$$\underline{\lim_{k\to 0}\delta_\ell^{(Born)}(k)} = \underbrace{-\frac{2m}{\hbar^2}\, 2^{2\ell}\left[\frac{\ell!}{(2\ell+1)!}\right]^2}_{\text{messy overall coeff.}} \; \underbrace{k^{2\ell+1}} \; \underbrace{\int_0^\infty r^{2\ell+2} V(r)\, dr}_{r^p \text{ moments of } V(r)}$$

threshold behavior of $k^{2\ell+1}$, as suggested prev.