Scattering and partial waves

Previously we derived the Born series,
which is most useful for weak potentials and high energies.

At low energies most of the scattering may take place at low angular momentum ℓ; for this reason it's useful to introduce a "partial-wave decomposition" of the scattering process. This is most useful for $V(r)$, a radial potential, in which case total angular momentum is conserved in the scattering process.

For a free particle we have two "standard" basis sets,

spherical Bessel function

$$\left\{ e^{i\vec{k}\cdot\vec{x}} \right\} \qquad\qquad \left\{ j_\ell(kr)\, Y_{\ell m}(\Omega) \right\}$$

plane waves — spherical waves

each satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

$(\hbar^2k^2/2m = E)$ and can be used to expand any soln of the free Schrödinger equation that's nonsingular.

(for singular solns at $r=0$ must include $\eta_\ell(kr)Y_{\ell m}(\Omega)$)

spherical Neumann function

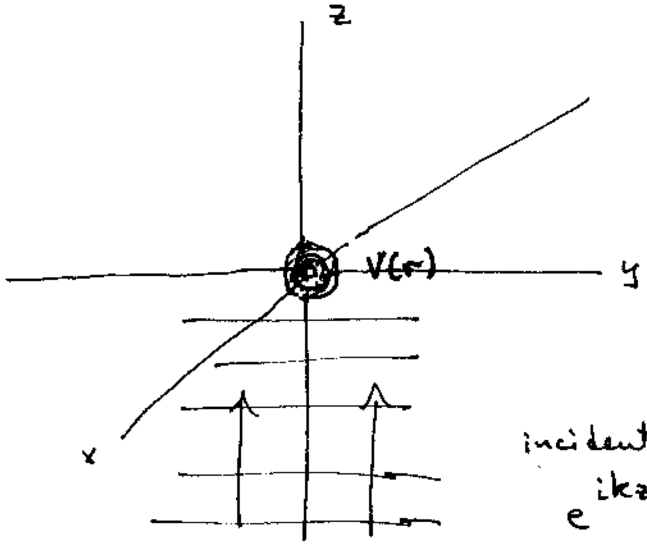
We could write one of either set as an expansion in the other set.

In particular, for the scattering problem, let's expand an incident plane wave $\underline{\psi_{inc} = e^{ikz}}$
in spherical waves:

m ≠ 0 not present because this gives φ - dependence

$$\psi_{inc} = \underbrace{e^{ikz}}_{e^{ikr\cos\theta}} = \sum_\ell \tilde{c}_\ell\, j_\ell(kr)\, Y_{\ell 0}(\Omega)$$

$$= \sum_\ell c_\ell\, j_\ell(kr)\, P_\ell(\cos\theta)$$



famous series ...
result is

$$c_\ell = (2\ell+1)\, i^\ell$$

incident plane wave

$$e^{ikz} = e^{ikr\cos\theta}$$

Proof of this expansion?

$$e^{ikr\mu} = \sum_{\ell=0}^{\infty} \underbrace{f_\ell(kr)}_{\substack{\text{coeff.}\\ \text{functions}}} P_\ell(\mu)$$

$\{P_\ell(\mu)\}$ are a complete set for $\mu \in [-1,1]$

$$\underbrace{\int_{-1}^{1} e^{ikr\mu} P_\ell(\mu)\, d\mu}_{2 i^\ell j_\ell(kr)} = \sum_{\ell'} f_{\ell'}(kr) \underbrace{\int_{-1}^{1} P_\ell(\mu) P_{\ell'}(\mu)\, d\mu}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

(a useful integral to know!)

$$= \frac{2}{2\ell+1} f_\ell(kr)$$

$$\therefore\ f_\ell(kr) = \underbrace{(2\ell+1)\, i^\ell}_{\text{our } c_\ell}\, j_\ell(kr)$$

That gives us the plane wave part in

$$\psi_{full} = \underbrace{e^{ikz}}_{\psi_{inc.}} + \underbrace{f(\Omega)\,\frac{e^{ikr}}{r}}_{\psi_{scat}}$$

what about a similar expansion of the full wfn?

The fact that the soln is not free $\forall\ \vec{x}$ (there is a potential) tells us that asymptotically (where $V=0$) $r\to\infty$, it will be a linear superposition of both the regular j_ℓ and the singular η_ℓ solution,

$$\psi_{full} = \sum_\ell \underbrace{R_\ell(kr)}\ \underbrace{P_\ell(\cos\theta)}$$

no $m\neq 0$ $Y_{\ell m}(\Omega)$ if this is a radial potential $V(r)$ doing the scattering.

linear combination $A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr)$

of the two indep. solns as $kr\to\infty$.

To see how this differs from the pure $V=0$ soln $j_\ell(kr)$ in the far-field (relevant for scat.), use

$$\lim_{x\to\infty} j_\ell(x) = \frac{1}{x}\sin\left(x - \ell\frac{\pi}{2}\right)$$

$$\lim_{x\to\infty} \eta_\ell(x) = -\frac{1}{x}\cos\left(x - \ell\frac{\pi}{2}\right)$$

(90° out of phase)

$j_0 = \frac{s}{x}$

$j_1 = \frac{s}{x^2} - \frac{c}{x}$

$j_2 = \left(\frac{3}{x^3} - \frac{1}{x}\right)s - \frac{3}{x^2}c$

$\eta_0 = -\frac{c}{x}$ $\quad \eta_1 = -\frac{c}{x^2} - \frac{s}{x}$

$\eta_2 = -\left(\frac{3}{x^3} - \frac{1}{x}\right)c - \frac{3}{x^2}s$

So having a linear combination of these can be written asymptotically as a shifted trig. function:

$$\lim_{kr\to\infty} A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr) = C_\ell \underbrace{\frac{1}{kr}\sin\left(kr - \ell\frac{\pi}{2}\right.}_{\text{pure } j_\ell \text{ part}} \left. + \delta_\ell\right)$$

"phase shift" of ℓ^{th} partial wave relative to the free $j_\ell(kr)$ solution.

where δ_ℓ is $\quad \delta_\ell = -\tan^{-1}\left(B_\ell / A_\ell\right)$

Thus we have expansions for ψ_{inc} and ψ_{full}, we can combine these to find the expansion coeffs for the full wavefunction, $\{a_\ell\}$ in terms of phase shifts $\{\delta_\ell\}$:

$$\psi_{full} = \sum_{\ell=0}^{\infty} a_\ell R_\ell(kr) P_\ell(\mu)$$

in terms of δ_ℓ. This will give a nice formula for $\frac{d\sigma}{d\Omega}$ in terms of phase shifts. + $f(\Omega)$ + σ

$$\lim_{kr\to\infty} \psi_{full} = \sum_\ell a_\ell P_\ell(\mu) \frac{\sin\left(kr - \ell\frac{\pi}{2} + \delta_\ell\right)}{kr}$$

each a mix of incoming & outgoing.

$$\shortparallel$$

$$\psi_{inc} = \sum_\ell (2\ell+1) i^\ell P_\ell(\mu) \frac{\sin\left(kr - \ell\frac{\pi}{2}\right)}{kr}$$

$$+$$

$$\psi_{scat} = \sum_\ell f(\Omega) \frac{e^{ikr}}{r} \quad \leftarrow \text{outgoing wave only}$$

equate coeffs of $\frac{e^{-ikr}}{kr} P_\ell(\mu)$: $\quad a_\ell e^{-i\delta_\ell} = (2\ell+1) i^\ell$

$$\underline{a_\ell = (2\ell+1) i^\ell e^{i\delta_\ell}}$$

Now we can find the coeffs of the scattered wave,

$$f(\Omega) = \sum_\ell \underset{\sim}{f_\ell} P_\ell(\mu)$$

equate coeffs of $e^{+ikr} P_\ell(\mu)$:

$$\underset{\substack{\downarrow \\ (2\ell+1) i^\ell e^{i\delta_\ell}}}{a_\ell} \cdot \frac{e^{i(kr - \ell\frac{\pi}{2} + \delta_\ell)}}{2i \cdot kr} = (2\ell+1) i^\ell \frac{e^{i(kr-\ell\frac{\pi}{2})}}{2i\, kr} + f_\ell \frac{e^{ikr}}{r}$$

coeff $\frac{e^{ikr}}{r}$:

$$f_\ell = \frac{1}{2ik}(2\ell+1)\underbrace{i^\ell e^{-i\ell\frac{\pi}{2}}}_{1}\left(e^{2i\delta_\ell} - 1\right)$$

$$= \frac{1}{2ik}(2\ell+1)\left(e^{2i\delta_\ell} - 1\right)$$

$$\therefore \; f_\ell = \frac{1}{k}(2\ell+1) e^{i\delta_\ell} \sin\delta_\ell$$

this gives us the partial-wave expansion for the scattering amplitude in terms of phase shifts,

$$\boxed{f(\Omega) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \cdot P_\ell(\mu)}$$

The differential cross section is as usual $\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$, so there are cross terms between the different "partial waves". However, in the <u>total</u> cross section there is <u>no</u> cross term:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \sum_{\ell\ell'} f_\ell f_{\ell'}^* \underbrace{\int d\Omega\, P_\ell(\mu) P_{\ell'}(\mu)}_{\underbrace{2\pi \int_{-1}^{1} d\mu\, P_\ell P_{\ell'}}_{\frac{4\pi}{(2\ell+1)}\delta_{\ell\ell'}}},$$

$$\sigma = \sum_{\ell=0}^{\infty} \frac{4\pi}{(2\ell+1)} |f_\ell|^2 = \sum_{\ell=0}^{\infty} \frac{4\pi}{k^2} (2\ell+1) \sin^2\delta_\ell$$

The neat result.

$$\equiv \sum_{\ell=0}^{\infty} \sigma_\ell$$

ℓ^{th} partial-wave cross section.

Note there is a maximum possible value for the cross section in each partial wave,

$$\sigma_\ell^{max} = \frac{4\pi(2\ell+1)}{k^2}$$

"unitarity bound"

This expansion is most useful at low energies.
Typically near threshold S-wave ($\ell=0$) scattering dominates, then we pick up contributions from higher ℓ with increasing energy (k). We will see examples...