Born order scat. examples

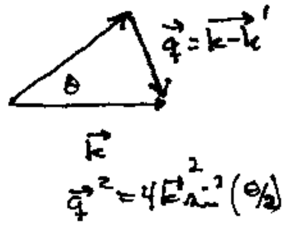
1. Coulomb scat. $\quad V(r) = -\dfrac{\alpha \hbar c}{r} = -\dfrac{e^2}{r}.$

$$f(\vec{q}) = -\frac{m}{2\pi\hbar^2}\int d^3x'\, V(\vec{x}')\, e^{i\vec{q}\cdot\vec{x}'}$$

we find some general simplifications for a $V(r$ only$)$:

$$f(\vec{q}) = \underset{\substack{\text{only}\\ \parallel \\ f(q)}}{f(|\vec{q}|)} = -\frac{m}{2\pi\hbar^2}\int_0^\infty r'^2 dr'\, V(r') \underbrace{\int d\Omega'\, e^{i\vec{q}\cdot\vec{x}'}}_{\substack{\text{can choose any}\\ \text{coord sys.}\\ \text{choose one in}\\ \text{which } \vec{q} \equiv q\hat{z}}}$$

$(q = |\vec{q}|)$



$$\longrightarrow \quad 2\pi \int_{-1}^{1} d\mu'\, e^{iqr'\mu'} = 2\pi\, \frac{e^{iqr'} - e^{-iqr'}}{iqr'} = \frac{4\pi}{qr'} \sin(qr')$$

$$\underline{f(q)} = \underline{-\frac{2m}{\hbar^2}\,\frac{1}{q}\int_0^\infty r\,dr\, V(r)\sin(qr)}$$

(r' is a dummy var. call it r)

Note for this integral to be defined, asymptotically as $\underline{r \to \infty}$, it should fall <u>faster</u> <u>than r^{-1}</u>, hence we have to 'screen' the Coulomb potential to get a sensible answer. Similarly at <u>small</u> r, $V(r)$ must diverge <u>more slowly</u> <u>than</u> r^{-2}. (faster than r^{-2} is unphysical – no normalizable bound states).

<u>screened</u> Coulomb

$$V(r) = -\frac{\alpha \hbar c}{r} e^{-\lambda r}$$

$$f(q) = \frac{2m\alpha c}{\hbar} \frac{1}{q} \underbrace{\int_0^\infty dr \sin(qr) e^{-\lambda r}}_{\frac{q}{q^2+\lambda^2}} = \frac{2m\alpha c}{\hbar} \frac{1}{(q^2+\lambda^2)}$$

∞ range Cou. limit is $\lambda \to 0$: $f(q) = + \frac{2m\alpha c}{\hbar} \frac{1}{q^2}$

$$\frac{d\sigma}{d\Omega} = |f(q)|^2 = \left(\frac{2m\alpha c}{\hbar}\right)^2 \frac{1}{16 \vec{k}^4 \sin^4(\theta/2)}$$

$$= \frac{m^2 \alpha^2 c^2}{4 \hbar^2 \vec{k}^4} \frac{1}{\sin^4(\theta/2)}$$

<u>extremely</u> forward peaked. $f \to \infty$ as $\theta \to 0$.

or since $\vec{p} = \hbar \vec{k}$

$$\frac{d\sigma}{d\Omega} = \frac{m^2 c^2 \hbar^2 \alpha^2}{4 \vec{p}^4} \frac{1}{\sin^4(\theta/2)}$$

<u>Rutherford</u> <u>cross section</u>

$$\frac{d\sigma}{d\Omega} = \frac{m^2 e^4}{4 \vec{p}^4} \frac{1}{\sin^4(\theta/2)}$$

since $e^2 = \hbar c \alpha$

units? $\left(\frac{mc}{\hbar k^2}\right)^2 = \left(\frac{mc^2}{\hbar c k^2}\right)^2 = \left[\frac{\text{erg}}{(\text{erg}\cdot\text{cm})\cdot\text{cm}^{-2}}\right]^2 = [\text{cm}^2]$ ✓

<u>total</u> Coulomb cross section

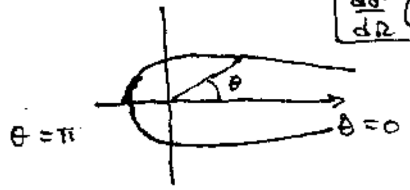
$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = \frac{m^2 e^4}{4\vec{p}^{\,4}} \cdot \underbrace{2\pi}_{\int_0^{2\pi} d\varphi} \cdot \int_{\theta=\pi\ (\text{back})}^{\theta=0\ (\text{forward})} \frac{\sin\theta \, d\theta}{\sin^4(\theta/2)}$$

This integral is divergent.

note at small angles $\sim \displaystyle\int^{\theta=0} \frac{\theta \, d\theta}{(\theta/2)^4} \ \tilde{\propto} \int^{\theta=0} \frac{d\theta}{\theta^3}$, <u>divergent.</u>

famous result: ∞ range potential $V \sim \frac{1}{r}$ implies that everything scatters, dominantly into small forward angles.

$\frac{d\sigma}{d\Omega}(\theta)$ (cou.)



Example of a finite cross section — <u>screened Coulomb</u> $V(r) = -\frac{\hbar c \alpha}{r} e^{-\lambda r}$ or $e^{-r/r_{screen}}$

(we already did most of this) $\lambda = \frac{1}{r_s}$

$$\frac{d\sigma}{d\Omega} = |f(\vec{q})|^2 = \left(\frac{2m\alpha c}{\hbar}\right)^2 \frac{1}{(\vec{q}^{\,2}+\lambda^2)^2}$$

$$= \underbrace{\frac{m^2 e^4}{4\vec{p}^{\,4}}}_{\left(\frac{\hbar m c \alpha}{2\vec{p}^{\,2}}\right)^2} \frac{1}{\left[\sin^2(\theta/2) + \underbrace{\frac{\lambda^2}{4\vec{k}^2}}_{\left(\frac{\hbar^2\lambda^2}{4\vec{p}^{\,2}}\right) = \left(\frac{\hbar^2}{4\vec{p}^{\,2} r_s^2}\right)}\right]^2}$$

The shape of $W(\theta) = \frac{d\sigma}{d\Omega}$ is qualitatively determined by the relative size of $\hbar/p = \lambda_{compton}$ and r_s.

If we rewrite in these variables,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar m c \alpha}{2\vec{p}^{\,2}}\right)^2 \cdot \frac{1}{\left[\sin^2(\theta/2) + \left(\lambda_{compt}/4\pi r_s\right)^2\right]^2}$$

ratio of forward to backscattered intensity is

$$\frac{W(0)}{W(\pi)} = 1 + \left(\frac{4\pi r_{scat.}}{\lambda_{compt.}}\right)^4$$

$r_{scat} \to 0$, $\frac{d\sigma}{d\Omega} \to \left(\frac{\hbar m c \alpha}{2\vec{p}^{\,2}}\right)^2 \cdot \left(\frac{4\pi r_{scat}}{\lambda_{compt}}\right)^4$ = <u>isotropic</u>,

"S-wave"

Small scatterers give larger scat. angle.

$r_s \to 0$, we recover <u>isotropic</u> scattering.

$r_{scat} \to \infty$, $\frac{W(0)}{W(\pi)} \to \infty$ (becomes forward-divergent Coulomb $\frac{d\sigma}{d\Omega}$ limit).

So, can we calculate the total cross section for the screened case?

$$\frac{d\sigma}{d\Omega} = \left(\frac{2m\alpha c}{\hbar}\right)^2 \left(\vec{q}^{\,2}+\lambda^2\right)^{-2}$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = 2\pi \int_{-1}^{1} d\mu \quad \frac{d\sigma}{d\Omega}(\mu)$$

note $\vec{q}^{\,2} = 2\vec{k}^2(1-\cos\theta) = 2\vec{k}^2(1-\mu)$,
so our integral can also be written as

$$\sigma = 2\pi \int_0^{4\vec{k}^2} \frac{dq^2}{2k^2} \quad \frac{d\sigma}{d\Omega}(q^2).$$

$$\underline{\sigma = \frac{\pi}{k^2} \int_0^{4k^2} \frac{d\sigma}{d\Omega}\, dq^2}$$

This case

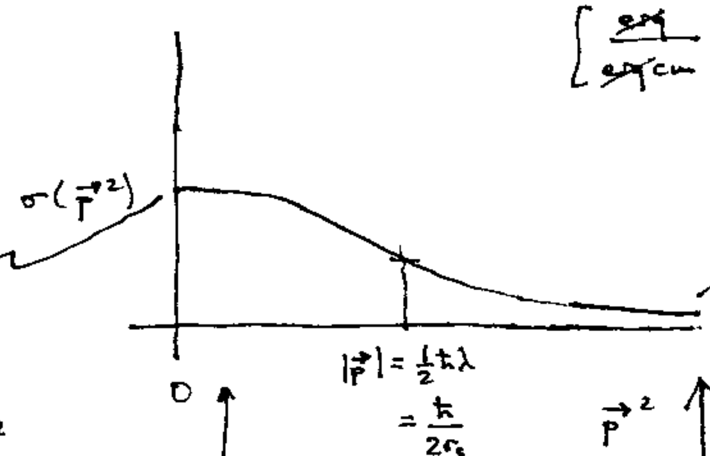
$$\sigma = \frac{\pi}{k^2} \int_0^{4k^2} \left(\frac{2m\alpha c}{\hbar}\right)^2 \frac{dq^2}{(q^2+\lambda^2)^2}$$

$$= \frac{\pi}{k^2} \left(\frac{2m\alpha c}{\hbar}\right)^2 \underbrace{\left\{ \frac{-1}{(q^2+\lambda^2)} \Bigg|_{q^2=0}^{q^2=4k^2} \right\}}_{\frac{1}{\lambda^2} - \frac{1}{(4k^2+\lambda^2)} = \frac{4k^2}{\lambda^2(4k^2+\lambda^2)}}$$

$$\underline{\sigma = \frac{16\pi}{\lambda^4} \left(\frac{m\alpha c}{\hbar}\right)^2 \left(1 + \frac{4k^2}{\lambda^2}\right)^{-1}} \longrightarrow \frac{4k^2}{\lambda^2} = \frac{4\vec{p}^{\,2}}{\hbar^2\lambda^2}$$

<u>units?</u>

$$[\mathrm{cm}^2] \stackrel{?}{=} [\mathrm{cm}^{+4}]\cdot\underbrace{\left(\frac{mc^2}{\hbar c}\right)^2}_{\left[\frac{\cancel{\mathrm{erg}}}{\cancel{\mathrm{erg}}\,\mathrm{cm}}\right]^2} = \mathrm{cm}^{+4}\cdot\mathrm{cm}^{-2} = \mathrm{cm}^2 \checkmark$$



low-energy cross section

$$\sigma_0 = \frac{16\pi}{\lambda^4}\left(\frac{m\alpha c}{\hbar}\right)^2$$

(diverges as $\lambda \to 0$, unscreened Coulomb)

high-energy limit

$$\sigma \approx \frac{4\pi\hbar^2}{\lambda^2}\cdot\frac{1}{\vec{p}^{\,2}}$$

(fall as $\frac{1}{E^p}$ is typical of h.e. limit. (p = some power like 1 or 2))

characteristic scale for decrease is set by the length scale $r_s = 1/\lambda$ in the potential relative to the Compton wavelength $h/|\vec{p}|$.

soft scat.

$\lambda_{\mathrm{Compt}} \gg r_{\mathrm{scat}}$.

$\frac{h}{|\vec{p}|} \gg r_{\mathrm{scat}}$,

$|\vec{p}| \ll h/r_{\mathrm{scat}}$

hard scat.

$\lambda_{\mathrm{Compt}} \ll r_{\mathrm{scat}}$.

$\frac{h}{|\vec{p}|} \ll r_{\mathrm{scat}}$.

$|\vec{p}| \gg h/r_{\mathrm{scat}}$