Path Integrals

Std. ref: R.P. Feynman & A. Hibbs, Path $\int$s and Quantum Mechanics

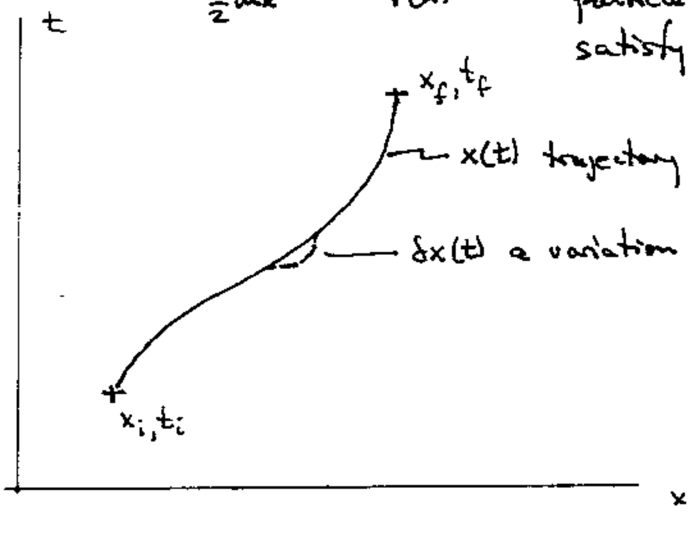
Basically path $\int$s are a very intuitive description of QM in terms of classical quantities — no operators. A recent development is the application to quantum field theory — lattice gauge theory simulations, especially of QCD. It's also a good way to resolve operator ordering ambiguities.

The idea is actually due to Dirac.

Dirac (according to Feynman) stated that $e^{iS/\hbar}$ is 'analogous to' the kernel of the Schrödinger equation, which Feynman began investigating

$$S = \int_{t_i}^{t_f} dt \left[\underset{\frac{1}{2}m\dot{x}^2}{T} - \underset{V(x)}{V} \right]$$

this is dependent on the trajectory $x(t)$ actually followed by the particle. classical paths satisfy $\delta S = 0$.



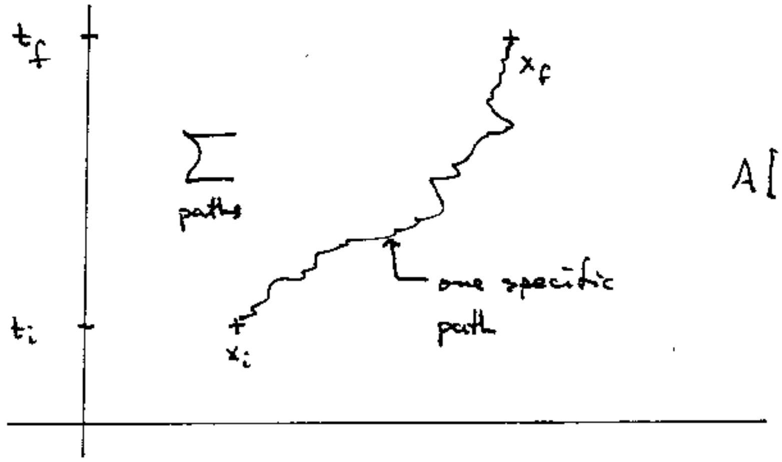
What is δS under the path variation $\delta x(t)$?

$$\delta S = \int dt \left[m\dot{x}\,\delta\dot{x} - V'(x)\,\delta x \right] = \int dt \underbrace{\left[-m\ddot{x} - V'(x) \right]}_{=0} \delta x(t) = 0 \qquad \forall\ \delta x(t)$$

or $ma = -V' = F$ ✓

$\delta S = 0$ was discovered by Euler, & claimed as a proof that God $\exists$ by Maupertuis.

In quantum mechanics, a particle that starts at (x_i, t_i) has an amplitude to finish at <u>any</u> (x_f, t_f) (for nonsingular V), hence we must allow more than a single path. Dirac / Feynman suggest that the particle actually follows <u>all</u> paths (non retracing in time), with an amplitude given by $e^{iS[\text{path}]/\hbar}$:



$$A[x_i, t_i \to x_f, t_f] = \sum_{\text{paths}} e^{iS[x(t)]/\hbar}$$

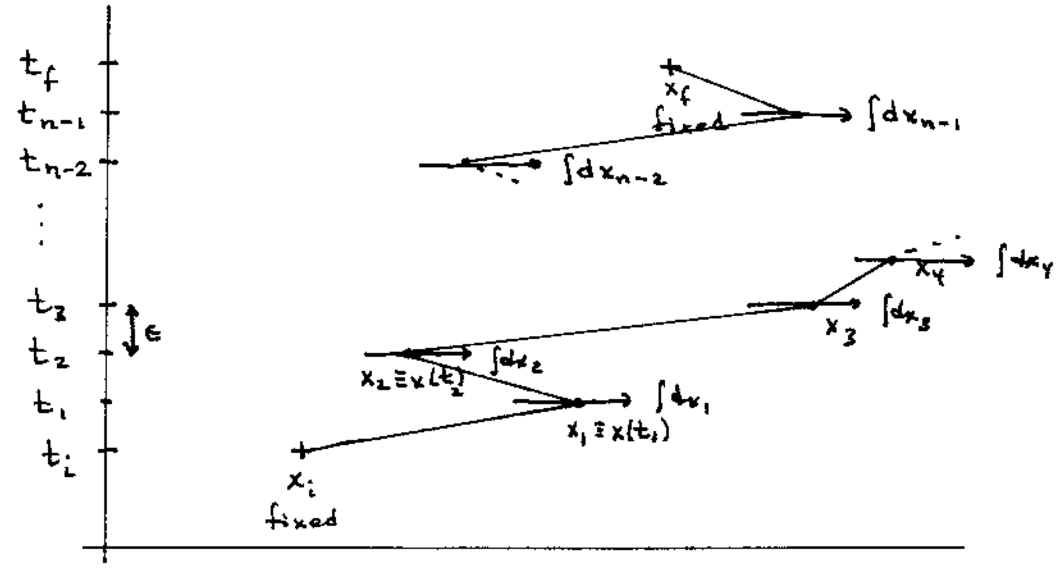
the path

How can we do a sum (integral) over all paths ?!

Ans: slice time and integrate over the value of x at each t_n

$\epsilon = \frac{t_f - t_i}{n}$

n "time slices"



"connect the dots"

integral over all dot positions gives all paths

The integral over all these points, as $\epsilon \to 0$, gives us an integral over paths

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots \int_{-\infty}^{\infty} dx_{n-1} \quad .$$

Clearly a dimensionful normalization will be required to take care of the length units. We associate one η with each integral:

$$\int \mathcal{D}x \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \eta\, dx_1 \int_{-\infty}^{\infty} \eta\, dx_2 \cdots \int_{-\infty}^{\infty} \eta\, dx_{n-1}$$

To actually do the integrals we have to discretize the action as well

$$\int_{t_i}^{t_f} dt\, S = \longrightarrow \int_{t_i}^{t_f} dt \left[\frac{1}{2} m \dot{x}^2 - V(x) \right]$$

$$\longrightarrow \sum_{\substack{i \\ \uparrow \\ \text{time slices}}} \underset{\substack{\uparrow \\ \Delta t}}{\epsilon} \left[\frac{1}{2} m \underbrace{\frac{(x_{i+1}-x_i)^2}{\epsilon^2}}_{\text{usual "forward-looking" time derivative}} - V(x_i) \right]$$

To get the norm η correct we can look at the free case $V=0$, over a single time slice. In this case, as $\epsilon \to 0$, our amplitude

$$A_{i\to f} = \sum_{\text{paths}} e^{i S_{\text{path}}/\hbar}$$

must approach a simple $\delta(x_i - x_f)$. Is this correct?

(and no path integral is necessary, although you find the same result by doing intermediate integrals.)

$$\lim_{\epsilon\to 0} \eta\, e^{\frac{im}{2\epsilon\hbar}(x_f-x_i)^2} \equiv \delta(x_f-x_i) \qquad \text{(specifies the path} \int \text{norm } \eta\text{)}$$

This is a famous "stationary phase" representation of a δ-function. The idea is that the phase varies slowly (quadratically) near $x_i = x_f$, but rapidly for $|x_f - x_i|$ large, so the function inside an integral acts increasingly like a δ-fn as $\epsilon \to 0$.

Test:

$$\int_{-\infty}^{\infty} \eta\, e^{\frac{im}{2\epsilon\hbar}(x-x')^2} \underbrace{f(x')}\, dx'$$

expand about $f(x)$: $f(x') = f(x) + (x'-x)\, f'(x) + \frac{1}{2}(x'-x)^2 f''(x) + \ldots$

change vars $y = x'-x$

$$= \int_{-\infty}^{\infty} \eta\, e^{\frac{im}{2\epsilon\hbar}y^2} \Big[f(x) + \underbrace{\cancel{y f'(x)}}_{\text{odd}}^{\,0} + \tfrac{1}{2} y^2 f''(x) + \ldots \Big]\, dy$$

$$= f(x) \cdot \eta \int_{-\infty}^{\infty} e^{\frac{im}{2\epsilon\hbar}y^2} dy + \tfrac{1}{2} f''(x) \cdot \eta \int_{-\infty}^{\infty} e^{\frac{im}{2\epsilon\hbar}y^2} y^2\, dy + \ldots$$

$\equiv f(x) +$ vanishing $\mathcal{O}(f''(x))$ contribution as $\epsilon \to 0$.

$a = \frac{m}{2i\epsilon\hbar}$ pretending this is ok

The stationary phase $\int$ is $\displaystyle\int_{-\infty}^{\infty} e^{-ay^2} dy = \sqrt{\frac{\pi}{a}} = \sqrt{\frac{2\pi i\epsilon\hbar}{m}}$

$$\therefore \quad \eta = \left[\frac{m}{2\pi i \epsilon\hbar} \right]^{1/2}$$

each $\int dx_i$ in the path $\int$ has <u>this</u> overall normalization associated with it.

The next-order term is

$$\frac{1}{2} f''(x) \underbrace{\eta \cdot \int_{-\infty}^{\infty} e^{-ay^2} y^2 dy}_{\frac{1}{2a} = \frac{i\epsilon\hbar}{m}} = \overset{\text{vanishes as } \epsilon \to 0}{\frac{i\epsilon\hbar}{2m}} f''(x)$$

Thus $\lim_{\epsilon \to 0} \left[\frac{m}{2\pi i \epsilon\hbar}\right]^{1/2} e^{\frac{im}{2\epsilon\hbar}(x_f - x_i)^2} = \delta(x_f - x_i)$ does act as a δ-function as $\epsilon \to 0$.

Consistency with the Schrödinger equation

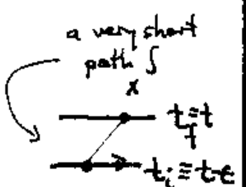
Assuming the path $\int$ is correct, so that the kernel of the time-dependent Schrödinger equation is

$$K(x_i, t_i \to x_f, t_f) = \int_{x_i, t_i}^{x_f, t_f} \mathcal{D}x \, e^{iS/\hbar} \qquad (\text{A}''')$$

and wavefunctions evolve as

$$\psi(x_f, t_f) = \int dx_i \, K(x_i, t_i \to x_f, t_f) \, \psi(x_i, t_i),$$

we may again use a single time slice to learn something interesting, the differential equation that this $\psi(x,t)$ must satisfy.



$$\psi(x,t) = \int dx_i \, \eta \, e^{\frac{i\epsilon}{\hbar}\left[\frac{m(x-x_i)^2}{2\epsilon^2} - V(x_i)\right]} \psi(x_i, t-\epsilon)$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx_i \, \eta \underbrace{e^{\frac{im(x-x_i)^2}{2\epsilon\hbar}}}_{\text{can't expand: rapidly osc. phase}} \left[1 - \frac{i\epsilon}{\hbar} V(x_i)\right] \underbrace{\psi(x_i, t-\epsilon)}_{[\psi(x_i,t) - \epsilon\dot{\psi}(x_i,t)] + O(\epsilon^2)}$$

Since the $e^{\frac{im(x-x_i)^2}{2\epsilon\hbar}}$ will again wash out contributions for large $|x-x_i|$, it's again useful to change variables to $y = x - x_i$ and expand in y.

$$V(x_i) = V(x-y) = V(x) - yV'(x) + \dots$$

$$\psi(x_i,t) = \psi(x-y,t) = \psi(x,t) - y\,\psi'(x,t) + \tfrac{1}{2}y^2\psi''(x,t) + \dots$$

Since $\int_{-\infty}^{\infty} dy\, e^{\frac{imy^2}{2\epsilon\hbar}}\,\eta\; y^p = 0$ for p odd, we need only keep even powers.

$$\underbrace{\int_{-\infty}^{\infty} dy\,\eta\, e^{\frac{imy^2}{2\epsilon\hbar}} = 1} \qquad\qquad \underbrace{\int_{-\infty}^{\infty} dy\,\eta\, e^{\frac{imy^2}{2\epsilon\hbar}} y^2 = \frac{i\epsilon\hbar}{m}}$$

Each power of y^2 gives a power of ϵ. To compare the leading $\mathcal{O}(\epsilon)$ terms in the evolution equation we therefore keep only

$$-\frac{i\epsilon}{\hbar}V(x_i) \rightarrow -\frac{i\epsilon}{\hbar}V(x) + \mathcal{O}(\epsilon^2)$$

$$-\epsilon\dot{\psi}(x_i,t) \rightarrow -\epsilon\dot{\psi}(x,t) + \mathcal{O}(\epsilon^2)$$

$\int$ to 0.

but

$$\psi(x_i,t) \rightarrow \underbrace{\psi(x,t)}_{\mathcal{O}(\epsilon^0)} - \cancel{y\psi'}(x,t) + \underbrace{\tfrac{1}{2}y^2\psi''(x,t)}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2)$$

and our evolution equation becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} dy\;\eta\, e^{\frac{im}{2\epsilon\hbar}y^2}\left[1 - \frac{i\epsilon}{\hbar}V(x)\right]\left[\psi(x,t) + \tfrac{1}{2}y^2\psi''(x,t) - \epsilon\dot{\psi}(x,t)\right] + \mathcal{O}(\epsilon^2)$$

The integrals required are above. This gives

$$\psi(x,t) = \psi(x,t) - \frac{i\epsilon}{\hbar} V(x)\psi(x,t) - \epsilon\dot{\psi}(x,t) + \frac{1}{2}\frac{i\epsilon\hbar}{m}\psi''(x,t) + \mathcal{O}(\epsilon^2)$$

Thus if the path $\int$ is correct, the wavefunction $\psi(x,t)$ must satisfy the differential equation

$$\underline{V(x)\,\psi(x,t) - \frac{1}{2}\frac{\hbar^2}{m}\psi''(x,t) = i\hbar\dot{\psi}(x,t)}$$

which is the Schrödinger equation.

Thus the path $\int$ definition of amplitudes is equivalent to requiring time evolution according to the Schrödinger equation.