Representations of SU(n) + basis vectors

For finite groups,

$$\sum_{i=1}^{N_c} d_i^2 = N$$

implied a finite number of IRs.

For continuous groups we have an infinite number of group elements, so we also have an infinite number of IRs.

However, as for S_n, we can build them up from smaller representations by using Young symmetrizers to generate product basis states of definite symmetry, which span IRs.

We can begin with the smallest ($\underset{\sim}{n}$-dim) irred. representation of the group SU(n), known as the <u>fundamental</u> rep, spanned by the basis

$$\underset{\substack{\downarrow \\ i=1,\dots,n}}{u_i} \longrightarrow u_{\boxed{i}} \longrightarrow \square$$

e.g for SU(2), basis is $\overset{u_1}{\overset{\shortparallel}{|+\rangle}}$ and $\overset{u_2}{\overset{\shortparallel}{|-\rangle}}$, and we can generate the rotation matrices using

$$\underbrace{U_R}_{\hookrightarrow\, e^{+\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}} \begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix} = \mathcal{D}^{(\underset{\sim}{2})}(R)^T \begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix}$$

+ as discussed previously, the generators are

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma}$$

for the $\underset{\sim}{2}$-rep.

Exponentiation thus gives us the 2D rotation matrices $\{\mathcal{D}^{(\underset{\sim}{2})}(R)\}$.

To get larger IRs we symmetrize product basis states in a now familiar fashion,

su(2): $\quad \square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$

$$\mathbf{2} \cdot \mathbf{2} = \mathbf{3} \oplus \mathbf{1} \qquad su(2)$$

and in n-dims, $\quad \mathbf{n} \otimes \mathbf{n} = \underbrace{\tfrac{1}{2}\, n(n+1)} \oplus \tfrac{1}{2}\, n(n-1) \qquad su(n)$

su(3): $\quad \mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$

$\begin{array}{c}\square\\[-4pt]\square\end{array}$ is also a 3-dim rep.

note the completely antisymmetric state of 3 labels $\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}$ must be $\underline{1}$'d, and combining $\begin{array}{c}\square\\[-4pt]\square\end{array}$ with $\underset{\mathbf{3}}{\square}$ include this.

Hence it's "complementary" to $\mathbf{3}$, and is called $\bar{\mathbf{3}}$.

Again in SU(3),

$$\square \otimes \square \otimes \square = \square\square\square \oplus \begin{array}{l}\square\square\\[-4pt]\square\end{array} \oplus \begin{array}{l}\square\square\\[-4pt]\square\end{array} \oplus \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

su(n) $\quad n \cdot n \cdot n = \tfrac{1}{6}\, n(n+1)(n+2) + 2 \cdot x + \tfrac{1}{6}\, n(n-1)(n-2)$

$$\therefore\ x = \tfrac{1}{3}\, n(n^2-1)$$

SU(3)

The famous application of SU(3) is in the quark model.
There are actually 2 SU(3) symmetries, rotations between quark type "*flavor*", an *approx.* global SU(3)

and local rotations between an internal label called "*color*", an *exact*, local SU(3).

We will consider the original *flavor* - SU(3) first.

There are 6 types of quarks

	1st gen		2nd gen		3rd gen	
	u	d	s	c	b	t
mass	~ 5-10 MeV		~150 MeV	~1.5 GeV	~5 GeV	~170 GeV
Q	$+\frac{2}{3}e$	$-\frac{1}{3}e$	$-\frac{1}{3}e$	$+\frac{2}{3}e$	$-\frac{1}{3}e$	$+\frac{2}{3}e$

$(e=|q_e|)$

these 3 lightest are often considered in isolation, since they are produced more easily than c, b, t.

The quark - gluon Lagrangian is

gluon color index, 1..8

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \underbrace{\sum_{i=1}^{N_f=6}\bar{\psi}_i\left(i\not{\partial} - m_i - g\frac{\lambda^a}{2}A^a_\mu\right)\psi_i}_{=\sum_{i=1}^{N_f}\bar{\psi}_i \not{D}\psi_i - m_i\bar{\psi}_i\psi_i}$$

If all masses were the same we would have a global SU(6) symmetry.
However the wide scatter of masses breaks this badly.
It's useful to instead simply to forget c, b, t and pretend the light masses m_u, m_d, m_s are equal. This restricted set of complex (unitary) rotations $\begin{bmatrix}\psi_u\\ \psi_d\\ \psi_s\end{bmatrix} \to U\cdot\begin{bmatrix}\psi_u\\ \psi_d\\ \psi_s\end{bmatrix}$ is an <u>(approx.) SU(3) symmetry.</u>

What value is this observation?

SU(3) IRs can be used to label the (approximately) degenerate states.

$$u_i = \begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix} \quad \text{“}\ydiagram{1}\text{”}$$

(neglect spin at first. this will give $|u_+\rangle, |u_-\rangle$ etc)

one quark states

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1}$$

$$3 \otimes 3 \qquad 6 \oplus \bar{3}$$

"flavor" sextet, triplet diquark states

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

three quark states

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

"baryons"

"flavor" decimet octet singlet

includes nucleons

Antiparticles transform according to the complementary Young tableaux.

$$\bar{u}_i = \begin{bmatrix} |\bar{u}\rangle \\ -|\bar{d}\rangle \\ -|\bar{s}\rangle \end{bmatrix} \quad \text{“}\ydiagram{1,1}\text{”}$$

$$\bar{3}$$

phases implied by G-parity (nontrivial, sorry)

Combine quark and antiquark,

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

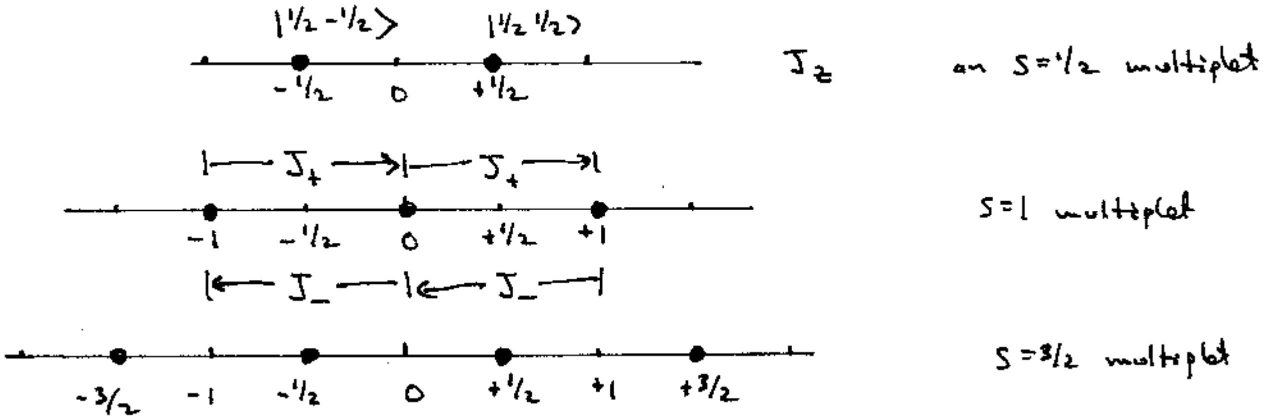
$$\bar{3} \otimes 3 = 8 \oplus 1$$

flavor octet singlet

and found that there was a maximum possible $m = +j$.
(similarly minimum $m = -j$)

Given these "stretched states", we can use repeated application of J_+ and J_- to construct all the states spanning an IR.

These may be conveniently displayed according to J_z eigenvalue on a "weight diagram":



etc

Incidentally, a polynomial $\mathcal{C}$ in the generators may be found that commutes with all the generators individually

$$[\mathcal{C}, J_i] = 0 \quad \forall\ i$$

but has a different eigenvalue on each IR, and hence may be used to label IRs. This is

$$\vec{J}^2 = \sum_i J_i^2$$

for SU(2). [Generically for a Lie algebra these are "Casimir operators", and $\vec{J}^2$ is the "quadratic Casimir".]

Labelling states

Through Schur's lemma, all states that span an IR of an unbroken symmetry are degenerate.

Another use of group theory (Lie groups) is that it specifies the properties of states <u>within</u> an IR.

Our classic example is $SU(2)$. The Lie algebra of $SU(2)$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

implies that one of these generators may be diagonalized (call it J_z), and we may form l.c.s of the other two that act as $\pm$ operators for the state's J_z eigenvalue:

$$J_\pm = J_x \pm i J_y \qquad\qquad J_- = J_+^\dagger$$

$$[J_z, J_\pm] = \pm J_\pm \equiv \pm \underbrace{\Delta m}_{\text{change in the eigenvalue of } J_z} J_\pm$$

proof: given $|jm\rangle$, $J_z|jm\rangle = m|jm\rangle$

$$J_z J_+ |jm\rangle - J_+ \underbrace{J_z |jm\rangle}_{m|jm\rangle} = +J_+|jm\rangle$$

$$\therefore\ J_z\{J_+|jm\rangle\} = (m+1)\{J_+|jm\rangle\}$$

by imposing comm rels such as $[J_+, J_-] = 2J_z$ we determined the coeffs of the $\pm$ operators, e.g.

$$J_+|jm\rangle = [j(j+1) - m(m+1)]^{1/2}|j\ m+1\rangle$$

SU(3)

For the larger Lie groups you may diagonalize >1 generator, the number that can be simultaneously diagonalized is n, where n is the rank of the algebra (A_n, B_n, C_n, D_n).

SU(N) has a rank $n = N-1$ Lie algebra, ∴ we can diagonalize $N-1$ generators simultaneously. The rank is also the number of independent polynomials in the generators that commute with all generators (≡ Casimir operators).
We <u>could</u> use their eigenvalues to label IRs. For SU(N), N>2, more often one simply uses the dimension of the IR as a generalized name.

For SU(3) there are $3^2-1=8$ generators. It's rank 2, so 2 can be diagonalized. In the standard "fundamental", 3-dim irrep of SU(3) the generator matrices ("Gell-Mann matrices" are)

$$\lambda_1 = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad \lambda_2 = \left[\begin{array}{cc|c} & -i & \\ i & & \\ \hline & & 0 \end{array}\right] \qquad \lambda_3 = \left[\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array}\right] \qquad \left(\vec{\lambda} = \left[\begin{array}{c|c} \vec{\sigma} & 0 \\ \hline 0 \; 0 & 0 \end{array}\right] \text{ for } 1,2,3 \right)$$

$$\lambda_4 = \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix} \qquad \lambda_5 = \begin{bmatrix} & & -i \\ & 0 & \\ i & & \end{bmatrix}$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda_7 = \begin{bmatrix} & & \\ & & -i \\ & i & \end{bmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$$

diagonal

$$J^a = \tfrac{1}{2}\lambda^a, \quad a = 1..8$$

↓ actually $\mathcal{D}^{(\underline{3})}(J^a)$

Note all are traceless.

$$1 = \mathrm{Det}\, e^{i\theta^a \cdot J^a} = e^{i\theta^a \cdot \underbrace{\mathrm{Tr}(J^a)}_{0}}$$

You can probably guess the 4x4 matrix representation for the 15 generators of SU(4).
[3 are diagonal.]

Note the set $\left\{ \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2} \right\}$ forms a subalgebra of SU(3), which is SU(2).

We can think of these as operating on a 2-d subspace of vectors spanned by an SU(2) basis

$$\left[\begin{array}{c|c} \boxed{\begin{matrix} SU(2) \\ \text{part} \end{matrix}} & \begin{matrix} \cdot \\ \cdot \end{matrix} \\ \cdot\ \cdot & \cdot \end{array}\right] \begin{bmatrix} |+\rangle \\ |-\rangle \\ - \end{bmatrix}$$

with a general basis vector

$$|\psi\rangle = c_+|+\rangle + c_-|-\rangle + \text{junk}$$

written as

$$|\psi\rangle = \begin{bmatrix} c_+ \\ c_- \\ \cdot \end{bmatrix} \leftarrow \text{not used for } SU(2)$$

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aside on quark flavor symmetry

Although we have always applied SU(2) to spins, it can be a symmetry of any system with two fermion labels (types).

e.g. for quarks, the u and d quarks have very similar masses. If they *were* identical masses & had identical couplings, operators that switched u- and d-quark type would be an exact symmetry of nature. (Obviously it's broken, since $m_u \neq m_d$ and $Q_u = +\frac{2}{3}e \neq Q_d = -\frac{1}{3}e$.)

$\uparrow$ $|Q_e|$

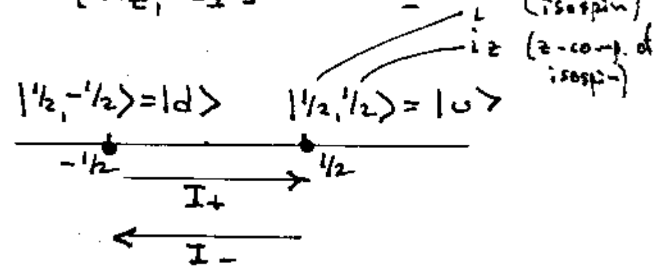
This good but not exact $u \rightleftarrows d$ symmetry is called "isospin", and the generators that raise & lower isospin (& z-comp.) also form an SU(2) Lie algebra

$$[I_i, I_j] = i\epsilon_{ijk} I_k$$

$$\therefore [I_z, I_\pm] = \pm I_\pm$$

isomorphic to the angular momentum SU(2) we have studied.

light quark isodoublet



$I_+|u\rangle = 0$, $I_+|d\rangle = |u\rangle$

$I_-|u\rangle = |d\rangle$, $I_-|d\rangle = 0$

$I_z|u\rangle = +\frac{1}{2}|u\rangle$, $I_z|d\rangle = -\frac{1}{2}|d\rangle$

You can construct larger multiplets like "diquarks" from $\frac{1}{2} \otimes \frac{1}{2}$

$$\underbrace{\begin{bmatrix} |uu\rangle \\ \frac{1}{\sqrt{2}}\left(|ud\rangle + |du\rangle\right) \\ |dd \end{bmatrix}}_{I=1 \text{ multiplet}} \qquad \underbrace{\frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right)}_{\substack{I=0 \\ \text{multiplet}}}$$

but these q, q^2 states are unphysical
(because of a different interaction, q-q color symmetry, which confusingly is also an SU(N) symm (N=3), but on a different degree of freedom - quark "$_{\text{color}}$".)

Color forces imply that only q^{3m} or $q^{m_1}\underbrace{\bar{q}^{m_2}}_{\text{count as } (q^2)^{m_2}}$
(with "triality zero") are physically allowed.

$m_1 + 2m_2 = 3m'$

Simplest are q^3 "baryons", made from (for isospin SU(2)):

$$q \otimes q \otimes q = \tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} = \tfrac{3}{2} \oplus \tfrac{1}{2} \oplus \tfrac{1}{2}$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1}$$

$$\underset{\sim}{4} \oplus \underset{\sim}{2} \oplus \underset{\sim}{2}$$

quartet doublet doublet under isospin.

e.g. of $\ydiagram{3}$, $|uuu\rangle = |3/2, 3/2\rangle$ "Δ^{++} baryon"

The less accurate SU(3) flavor symmetry follows if we bring in the 3rd "strange" quark $|s\rangle$. (Less accurate because $m_s \approx 150$ MeV $\gg m_{u,d} \approx 5$-10 MeV.) n.b. $Q_s = -\frac{1}{3}e$.

The generator ± ops that switch $u \rightleftarrows s$ and $d \rightleftarrows s$ analogously to $I_\pm$ are called $V_\pm$ and $U_\pm$;

$$I_\pm = I_x \pm i I_y \to \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm \longrightarrow \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm \longrightarrow \tfrac{1}{2}(\lambda_6 \pm i\lambda_7) \;;$$

on $\begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix}$ basis states;

n.b. this is standard for <u>any</u> rank-n Lie algebra: diagonalize n generators, and the remainder can be written as ± ops on these diagonal generators.

$$V_+|s\rangle = |u\rangle \qquad V_-|u\rangle = |s\rangle$$

$$U_+|s\rangle = |d\rangle \qquad U_-|d\rangle = |s\rangle$$

Note $$I_3 \begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}\begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix} = \begin{bmatrix} +\frac{1}{2}|u\rangle \\ -\frac{1}{2}|d\rangle \\ 0 \end{bmatrix}$$

$I_+|d\rangle = |u\rangle$
$I_-|u\rangle = |d\rangle$

$\therefore$ $(|u\rangle, |d\rangle)$ form an isodoublet

$$\underset{|d\rangle}{-\tfrac{1}{2}} \quad | \quad \underset{|u\rangle}{\tfrac{1}{2}} \quad I_3$$

The other diagonal generator is λ_8

$$\frac{1}{2}\lambda_8 \begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix} = \begin{bmatrix} +\frac{1}{2\sqrt{3}}|u\rangle \\ +\frac{1}{2\sqrt{3}}|d\rangle \\ -\frac{1}{\sqrt{3}}|s\rangle \end{bmatrix}$$

$\therefore$ after a shift & rescaling it counts strange quarks

not surprisingly this is <u>not</u> the historical normalization.

Instead it's "hypercharge" $$Y = \frac{2}{\sqrt{3}}\lambda_8 = \begin{bmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & -\frac{2}{3} \end{bmatrix} = \underset{\frac{1}{3}\text{ any } q}{B} + \underset{-n_s}{S}$$

An SU(3) weight diagram displays states that span an IR on a 2-d plot of their diagonal generator eigenvalues.
↑
rank n=2

These would be $\frac{1}{2}\lambda_3 = I_z$ and $\frac{1}{2}\lambda_8$, but for historical reasons

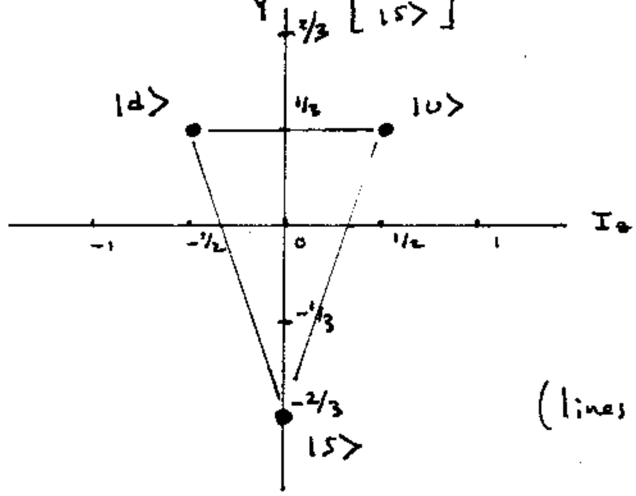
$$Y = B + S = \frac{2}{\sqrt{3}}\lambda_8$$

is used as the $2^{\underline{nd}}$ coordinate.

For the fundamental quark 'triplet' this weight diagram is

$$\begin{bmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{bmatrix}$$

$\underset{\sim}{3}$ of SU(3)



(lines only drawn for visualization)

Note since $Q_d = Q_s = -\frac{1}{3}$ and $Q_u = \frac{2}{3}$, and

$$Q = I_z + \frac{1}{2}Y$$

Since I_z & Y are additive quantum numbers this is true for any hadron. It was discovered before the quark model, and was just an observation relating strange & nonstrange hadrons.

You can build up weight diagrams for larger multiplets by starting with a "stretched" state like $|uuu\rangle$ and repeatedly applying I_-, V_-.

The "outer product" forms

$$I_- = |d\rangle\langle u|$$

$$V_- = |s\rangle\langle u|$$

$$U_- = |s\rangle\langle d| \qquad \text{etc}$$

are very useful for this.

n.b. $I_z = \frac{1}{2}\left(|u\rangle\langle u| - |d\rangle\langle d| \right)$ $\qquad I_+ = I_-^\dagger = |u\rangle\langle d|$ etc.

$$Y = \frac{1}{3}\left(|u\rangle\langle u| + |d\rangle\langle d| - 2|s\rangle\langle s| \right)$$

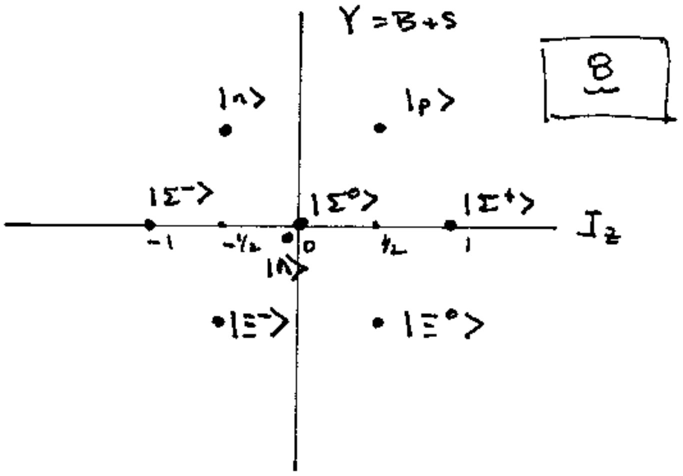
The classic example of an SU(3) rep. in the quark model is the baryon octet.

Starting with $|p\rangle \propto 2|uud\rangle - |udu\rangle - |duu\rangle$

we can operate with I_- to get

$$|n\rangle \propto 2|ddu\rangle - |dud\rangle - |udd\rangle$$

and so forth to fill out an $\underset{\sim}{8}$ of states. These are



uud, udd

uus, uds, dds

uss, dss

Note the presence of several SU(2) isospin IRs in this single SU(3) IR:

SU(3) $\qquad$ SU(2) subgroup

$$\underset{\sim}{8} = \underset{\sim}{3} + \underset{\sim}{2} + \underset{\sim}{2} + \underset{\sim}{1}$$