Continuous groups & Lie Groups

Many symmetry groups encountered in physics have a continuous infinity of elements.

e.g. translations $\quad U_{\vec{a}}\,\psi(\vec{x}) = \psi(T_a^{-1}\vec{x}) = \psi(\vec{x}-\vec{a})$

$$U_{\vec{a}} U_{\vec{b}} = U_{\vec{a}+\vec{b}} \qquad \text{product (closure)}$$

$$(U_a U_b) U_c = U_a (U_b U_c) \qquad \text{associativity}$$

$$U_{\vec{0}} = e \qquad \text{identity}$$

$$U_{\vec{a}} U_{-\vec{a}} = U_{-\vec{a}} U_{\vec{a}} = e \qquad \text{inverse}$$

e.g. 1D rotations $\quad U_{\varphi_1}\,\psi(\varphi) = \psi(\varphi - \varphi_1)$

Note both these examples are Abelian groups, so they have only 1D irreducible reps.

We noted previously that both these groups are of the form

$$\underset{\substack{\uparrow \\ \text{group element} \\ \text{label}}}{U_\theta} = e^{i\theta J} \leftarrow \text{generator} \qquad \theta = 0 \rightarrow \underset{\text{identity element}}{U_0 = I}$$

& for norm-preserving operations (typical for symmetry groups)

$$\underbrace{U^\dagger = U^{-1}}_{\text{unitary op.}} \;,\quad \text{or} \quad \underbrace{J^\dagger = J}_{\text{Hermetian generator}}$$

$$\text{unitary op.} \Rightarrow \text{Hermetian generator}$$

previous examples were

J_z (generator of rotations about the z-axis) $\underline{U_\varphi = e^{+\frac{i\varphi J_z}{\hbar}}}$

$\vec{P}$ (generator(s) of translations)

$$\underline{U_{\vec{a}} = e^{i\frac{\vec{a}\cdot\vec{P}}{\hbar}}}$$

a group that is made of exponentiated generators is known as a Lie group
linear combinations of Hermitian

$$U(\{\theta_a\}) \equiv e^{i\sum_{a=1}^{n}\theta^a J_a}$$

$\uparrow$ n real parameters

(the sign is a convention, we could take $\{\theta\}\to\{-\theta\}$ equally well.)

$$\approx \underbrace{1 + i\sum_{a=1}^{n}\theta^a J_a}_{\text{infinitesimal form.}}$$

The group axioms of closure and associativity constrain the properties of the generators $\{J_a\}$.

closure : $U_1 U$ must be a group element, as must $U_1 U U_1^{-1}$.

For infinitesimal $\{\theta^a\}$ in U this says

$$e^{i\theta_1\cdot J}\left(1 + i\sum_a \theta^a J^a\right) e^{-i\theta_1\cdot J}$$

$$= 1 + i\sum_a \theta^a\left\{J^a + i[\theta_1\cdot J, J^a] + \frac{i^2}{2!}[\theta_1\cdot J,[\theta_1\cdot J, J^a]] + \dots\right\}$$

$$= 1 + i \sum_a \theta^a \left\{ J^a + i \sum_b \theta_{1b} [J^b, J^a] + \frac{i^2}{2!} \sum_{bb'} \theta_{1b}\theta_{1b'} [J^{b'}[J^b, J^a]] + \dots \right\}$$

$$\equiv 1 + i \sum_a \theta'^a J^a \quad \text{if } U_1 U U_1^{-1}$$ is also in the group (must also be close to the identity)

This requires that the commutator $[J^b, J^a]$ not give any new operators, but instead simply returns a linear combination of the same generators $\{J^a\}$ we started with.

(If true for $[J^b, J^a]$, it will also be true for $[J^{b''}, [J^{b'}, \dots, J^a]]\dots]$ nested commutators.)

Thus, closure requires

$\sum_{c=1}^{n}$ implicit

$$\boxed{[J^a, J^b] = i \underbrace{f^{abc}}_{\text{"structure coefficients" that define the group}} J^c}$$

This set of commutations is called a Lie algebra and defines the Lie group.

The associativity property also implies a relation between generators,

$$[J^a, [J^b, J^c]] + \text{et. cyc.} = 0$$

"Jacobi identity"

How many sets of generators satisfy this definition of a Lie algebra?

There are 4 general series of solutions A_n, B_n, C_n, D_n and a few "exceptional" Lie algebras

$$G_2, F_4, E_6, E_7, E_8.$$

We will only discuss the general series, and only in terms of their invariants rather than the algebras themselves.

<u>Invariants</u> of Lie groups.

The groups generated by the Lie algebras in the general series have a very simple interpretation in terms of invariants.

One e.g., 3D rotations.

We can define rotations as the set of (real) linear transformations on a 3-vector $\vec{x}$ that leave the length of the vector

$$\vec{x}^2 = x_1^2 + x_2^2 + x_3^2$$

invariant

$$x_i' = \underbrace{R_{ij}}_{\text{a (real) rotation matrix}} x_j$$

det = +1 — "special"

$R^T = R^{-1}$ — "orthogonal"

} $\underset{\sim}{SO}$

operates on a 3-vector → $\underbrace{SO(3)}$ name of the (real) rotation group in 3D.

These 3×3 matrices may be generated by

$$R = e^{i\sum_{a=1}^{3} \theta^a J^a} = \mathcal{D}^{(3)}(\{\theta^a\})$$

$\underbrace{J_x, J_y, J_z}_{(3\times 3,\ \text{"spin-1"},\ \text{rep})}$ = Generators of rotations.

$$[J_i, J_j] = i\,\underbrace{\epsilon_{ijk}}\,J_k$$

"structure coefficients" of the SO(3) Lie algebra

There are $\underline{3}$ generators because in 3D there are $\frac{1}{2}d(d-1) = \frac{1}{2}\cdot 3\cdot 2 = 3$ indep rotation planes.

Rotations in higher dimensions:

$$\left.\begin{array}{lll} & \underline{SO(n)} & (n \text{ even}) \\ \text{and} & \underline{SO(n)} & (n \text{ odd}) \end{array}\right\} \text{actually their Lie algebras, with } \tfrac{1}{2}n(n-1) \text{ generators}$$

are 2 of the 4 general series of Lie algebras.

[Actually $SO(3) \subset SU(2)$ and $SO(4) = SU(2) \otimes SU(2)$ are special cases.]

Complex linear transformations:

$$z_i' = \underbrace{U_{ij}}_{\substack{\text{a complex, unitary} \\ \text{matrix}}} z_j$$

$\left.\begin{array}{ll} \det U = +1 \rightarrow & \text{"special"} \\ U^{+} = U^{-1} & \text{"unitary"} \end{array}\right\} SU$

operates on an n-vector $(z_1, \ldots z_n)$ $\longrightarrow$ $\underline{\underline{SU(n)}}$

These transformations preserve the complex norm

$$|\vec{z}|^2 = |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2$$

$\#$ params. $= \#$ generators $= \underline{n^2 - 1}$

$SU(2) \rightarrow 2^2 - 1 = 3$ generators. Same algebra as $SO(3)$

but the matrices start smaller! $\quad [J^a, J^b] = i\epsilon_{abc} J^c$

"spin-½ rep" $J^a = \frac{1}{2}\sigma^a$ generates $\underline{2}$ rep of $SU(2)$.

$(\hbar = 1)$

Final general series are the "symplectic group" $SP(2n)$ that preserve the skew form

$$x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + \dots \; .$$

(This group sees rather fewer applications than $SO(n)$ or (especially) $\underline{SU(n)}$.)

Origin of SU(n) invariance

In quantum field theories such as QED or QCD that collectively describe the standard model, the $\underbrace{\text{fundamental fermion fields}}_{\text{leptons \& quarks}}$ $\underbrace{\text{\& bosons}}_{\gamma,\text{ gluons, W, Z bosons}}$ that interacte lead to

fairly simple fundamental interaction rules between particles, once we specify a Lorentz-invariant functional of the fields known as the action, S.

S is a space-time integral of a Lorentz invariant $\mathcal{L}$, known as the action density.

e.g. for photons it is

photon quantum field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - J_\mu A_\mu$$

electromagnetic 4-current $(\rho, \frac{1}{c}\vec{J})$

J_μ is actually constructed from Dirac fermion quantum fields ψ, as

$$J_\mu(\vec{x},t) = -e\, \psi_e^\dagger \underbrace{\gamma^0 \gamma_\mu}_{\substack{4\times4 \\ \text{Dirac} \\ \text{matrices}}} \psi_e \equiv -e\, \bar{\psi}_e \gamma_\mu \psi_e$$

it's adjoint

a 4-component Dirac field ($e \equiv$ electron)

Including mass terms,

$$\not{V} \equiv \gamma_\mu V_\mu$$

set of 4 4×4 "Dirac matrices"; any 4-vector

$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}}_{\text{free photon } \mathcal{L}} + \underbrace{\underbrace{\bar{\psi}_e(i\not{\partial}-m_e)\psi_e}_{\text{free electron } \mathcal{L}} \underbrace{- J_\mu^e A_\mu}_{J\cdot A \text{ minimal coupling}}}_{\bar{\psi}_e(i\not{\partial}-m_e-e\not{A})\psi_e}$$

In practice there are 3 distinct leptons that differ in their masses,

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \sum_{i=1}^{3}\bar{\psi}_i(i\not{\partial}-m_i-e\not{A})\psi_i$$

each has charge $-e$

$m_e \ll m_\mu \ll m_\tau$

$i = e^-, \mu^-, \tau^-$

However... suppose their masses <u>were</u> identical. Then

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \sum_{i=1}^{3}\bar{\psi}_i(i\not{\partial}-m-e\not{A})\psi_i$$

Note the quadratic form in the fermion fields

is generically of the form $\displaystyle\sum_{i=1}^{3}\psi_i^\dagger \mathcal{O}\psi_i$

and is invariant under a unitary transformation between the 3 types of fields, $\psi_i \to U_{ij}\psi_j \qquad \psi_i^\dagger \to \psi_K^\dagger U_{Ki}^\dagger$

$$\sum_{i=1}^{3}\psi_i^\dagger\mathcal{O}\psi_i \to \sum_{i,j,k}\psi_k^\dagger \underbrace{U_{ki}^\dagger U_{ij}}_{\substack{(U^\dagger U)_{kj} \\ =\delta_{kj}}}\mathcal{O}\psi_j = \sum_{j=1}^{3}\psi_j^\dagger\mathcal{O}\psi_j$$

constants, gradients, $f(x)$ etc. operates on different labels than the lepton type i

$\therefore$ <u>invariant</u>

This complex 3×3 unitary transformation is just SU(n) for n=3.

$$\therefore \quad \sum_{i=1}^{3} \psi_i^\dagger (i\not{\partial} - m - e\not{A}) \psi_i \qquad \text{is} \quad \underline{\text{SU(3) invariant}}.$$

We actually call this a "global SU(3) symmetry" because the transformation matrix U_{ij} has constant entries. If we generalize this to a <u>local</u> symmetry

$$\psi_i(x) \longrightarrow U_{ij}(x)\,\psi_j(x) \qquad \text{(local gauge invariance)}$$

this leads to the introduction of a generalized vector potential A_μ^a and to "non-Abelian gauge theories."

n^2-1 comps in SU(n).

examples are SU(3) color symm (QCD)
$A_\mu^a \to$ "gluons"

SU(2) ⊗ U(1) electroweak (Weinberg-Salam)
$A_\mu^a \to \gamma,\ W^\pm,\ Z^0$
(electroweak bosons + γ)

that's all the standard model has – non-Abelian and Abelian gauge theories
SU(3) ⊗ SU(2) ⊗ U(1).