Construction of symmetrized basis vectors – Young tableaux – and thence representation matrices.
(These are used both for construction of S_N IRs and IRs of continuous groups like the rotation group (spin states). SU(2)

In our example we used the 3 basis states

$|++-\rangle$, $|+-+\rangle$, $|-++\rangle$

and simply noted that the basis states

$$|\underline{1}\rangle = \frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle\right)$$

and the pair

$$|\underline{2}\rangle_1 = \frac{1}{\sqrt{3}}\left(|++-\rangle + \eta|+-+\rangle + \eta^2|-++\rangle\right)$$

and

$$|\underline{2}\rangle_2 = \frac{1}{\sqrt{3}}\left(|++-\rangle + \underbrace{\eta^2}_{\eta^*}|+-+\rangle + \underbrace{\eta^4}_{\eta^{*2}=\eta}|-++\rangle\right) \qquad \eta = e^{2\pi i/3} \quad (\eta^3=1,\ \eta^2=\eta^*)$$

spanned the $\underline{1} \oplus \underline{2}$ irreducible reps contained in the original reducible $\underset{\sim}{3}$ representation.

There is a systematic procedure for constructing these basis states, using operators known as "Young symmetrizers" that act on an initial, unsymmetrized state. These are constructed from diagrams or patterns knowns as "Young tableaux", that provide a convenient way to label the IRs.

A "Young tableau" for a state $|s_1 \ldots s_N\rangle$ with N objects (slots, lattice sites, etc) is a pattern of N boxes that shows what symmetry we wish to impose on the state. [Note for S_N, all those labels are different. Object 1…N.]

e.g. for $N=2$, rotation group, the states $\{|s_1 s_2\rangle\}$ with $s_{1,2} = \pm 1/2$ (e.g.) can be made symmetric □□ or antisymmetric ⊟. [States $|++\rangle$, $|--\rangle$ with identical labels allowed.]

The symmetrizer P corresponding to □□, which can only operate on the slots 1 & 2 since these are our only locations in $|s_1 s_2\rangle$, are on a bigger basis vector like $|s_1 s_2 s_3\rangle$ this could be $\boxed{1}\boxed{2}$, $\boxed{1}\boxed{3}$ or $\boxed{2}\boxed{3}$

$$\widetilde{\boxed{1}\boxed{2}} = e + (12)$$

applied to a general state $|s_1 s_2\rangle$ this gives us the symmetric combination

$$\boxed{1}\boxed{2}\,|s_1,s_2\rangle = \left\{ e + (12) \right\} |s_1,s_2\rangle = |s_1,s_2\rangle + |s_2,s_1\rangle$$

for all possible states with $s_1^{tot} = s_2^{tot} = 1/2$, this operator gives

$$\boxed{1}\boxed{2}\,|++\rangle = 2|++\rangle$$

$$\boxed{1}\boxed{2}\,|+-\rangle = |+-\rangle + |-+\rangle$$

$$\boxed{1}\boxed{2}\,|-+\rangle = |++\rangle + |+-\rangle = \text{''}$$

$$\boxed{1}\boxed{2}\,|--\rangle = 2|--\rangle$$

Similarly we may construct an antisymmetrizer

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} = e - (12)$$

which gives

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\,|++\rangle = 0 = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\,|--\rangle$$

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\,|+-\rangle = |+-\rangle - |-+\rangle = -\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\,|-+\rangle\right)$$

The sets of <u>two</u> spin-1/2 states with these distinct symmetries are

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\quad \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \qquad\qquad \boxed{}\boxed{}\left\{\begin{array}{c} |++\rangle \\ \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |--\rangle \end{array}\right\}$$

Aha! Recall the Clebsch-Gordon problem for two spins?

$$\underbrace{1/2 \otimes 1/2 = 1 \oplus 0}_{\text{spins}} \qquad \text{or} \qquad \underbrace{\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}}_{\text{dimensions}}$$

In which we combined two $s = 1/2$ state sets $(|+\rangle, |-\rangle) \otimes (|+\rangle, |-\rangle)$ and made states with net total spin,

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \qquad |1,m\rangle = \begin{cases} |++\rangle & m=+1 \\ \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) & m=0 \\ |--\rangle & m=-1 \end{cases}$$

evidently the operation of symmetrization or antisymmetrization (□□ or ⊟) of states has projected out basis vectors that span the two irreducible representations of the <u>rotation group</u>:

All within SU(2).

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$$

$$\underbrace{\underset{\substack{\text{one index} \\ (\text{here } S_z^1)}}{\square} \otimes \underset{\substack{2^{\text{nd}}\text{ index} \\ (\text{here } S_z^2)}}{\square}}_{\text{combining two smaller IRs of the rotation group}} = \underbrace{\underset{\text{symm}}{\square\square} \oplus \underset{\text{a-tisym}}{\boxminus}}_{\text{new IRs of the rotation group.}}$$

or $\underset{\sim}{2} \otimes \underset{\sim}{2} = \underline{3} \oplus \underline{1}$

[alternatively we are making up the 2 IRs of S_2 □□ and ⊟ ; from the IR □ of S_1.]

we are doing two problems concurrently here. 1^{st} lets do SU(2), will then return to S_N IRs.

What we are doing is to start with a 2-dim. irrep. of the rotation group, with basis vectors $\begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix}$, generically u_i (i: + or −), generically □

and looking at the transformation property of the product states

$$\underset{\square\otimes\square}{u_i v_j}$$

under rotations. Since symmetry / antisymmetry is preserved under SU(2) (complex rotations), the basis states that span the product space □⊗□ can be rewritten as the symm/asym lin. combinations

$$(u_i v_j + u_j v_i)/\sqrt{2} \qquad (\square\square \text{ in shorthand})$$

$$\text{and} \quad (u_i v_j - u_j v_i)/\sqrt{2} \qquad (\boxminus \text{ in shorthand})$$

that will only rotate into themselves.

The tableaux $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are used generically to refer both to the basis states that span an IR and to the representation matrices themselves.

With ≥ 3 objects, we can also have states of mixed symmetry.
slots, not values of state labels in the slot.

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$$

$$\square \otimes \square = \underbrace{\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\text{two IRs from the product of two "fundamental" reps.}}$$

relevant to $|s_1^z, s_2^z\rangle$ states

adjoin a third object

$$\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} =$$

$$\square \otimes \square \otimes \square = \left(\square\square \otimes \square\right) \oplus \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \square\right)$$

relevant to $|s_1^z s_2^z s_3^z\rangle$ states

$$= \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$$

$$\tfrac{3}{2} \oplus \tfrac{1}{2} \oplus \tfrac{1}{2} \oplus 0$$

$\nexists$ tot. antisym for 3 spin-1/2 objects

<u>The Young symmetrizers in general are</u>

$$Y = QP$$

$$P = \sum_p p \qquad Q = \sum_q (-)^q q$$

p: horizontal (symm) perms; q: vertical (antisymm) perms; $(-)^q$: +/− 1 if even/odd

$$\boxed{1|2|3} = e + (12) + (13) + (23) + (123) + (132)$$

gives ~~the~~ a totally symmetric state, <u>unique</u>, when operating on $|abc\rangle$ — if $a \neq b \neq c$ (S_3 problem) (spans $\underline{1}$ triv. rep.)

$$\begin{smallmatrix}1\\2\\3\end{smallmatrix} = e - (12) - (13) - (23) + (123) + (132)$$

gives ~~the~~ a totally antisymmetric state (also unique) when op. on $|abc\rangle$, $a \neq b \neq c$ (spans $\underline{1}'$, alt. rep)

Using our previous example of the 3 states $|++-)$, $|+-+)$, $|-++)$, let's see what the symmetrizers do

$$\boxed{1}\boxed{2}\boxed{3}\ |++-) = \left[e + (12) + (13) + (23) + (123) + (132)\right] |++-)$$

$$= 2|++-) + 2|-++) + 2|+-+) = 2\sqrt{3}\, |\psi_0\rangle$$

$$\boxed{|\psi_0) = \frac{1}{\sqrt{3}}\left(|++-) + |+-+) + |-++)\right)}$$

→ is the unique state that spans the trivial $\underline{1}$ rep of S_3.

(same result operating on $|+-+)$ or $|-++)$
→ only 1 state in the totally symm $\boxed{\ }\boxed{\ }\boxed{\ }$ rep from our set)

similarly

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ |++-) = \cancel{|++-)} - \cancel{|++-)} - \cancel{|-++)} - \cancel{|+-+)} + \cancel{|-++)} + \cancel{|+-+)}$$

$$= 0$$

these would be alternating rep. $\underline{1}'$ basis states.

as we found prev, there is no $\underline{1}'$ rep. present in our reducible $\underline{3}$.

The remaining states transform according to the mixed symmetry operators $\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array}$.

Let's see what basis states these produce.

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = \left[e-(13)\right]\left[e+(12)\right]$$

the $\begin{array}{|c|c|}\hline \, & \, \\\hline \, \\\cline{1-1}\end{array}$ tableaux both give mixed symm states, which are not unique. There are 2 independent linear combinations that survive this operation.
e.g. try on our original states $|++-\rangle$, $|+-+\rangle$, $|-++\rangle$

$$\left[e+(12)\right]|++-\rangle = 2|++-\rangle$$

$$\left[e-(13)\right]\left[e+(12)\right]|++-\rangle = 2\left(|++-\rangle - |-++\rangle\right) = 2\sqrt{2}\,|\psi_1\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|++-\rangle - |-++\rangle\right)$$

$$\left[e+(12)\right]|+-+\rangle = |+-+\rangle + |-++\rangle$$

$$\left[e-(13)\right]\left[e+(12)\right]|+-+\rangle = \cancel{|+-+\rangle} - \cancel{|+-+\rangle} + |-++\rangle - |++-\rangle = -\sqrt{2}\,|\psi_1\rangle$$

$$\left[e+(12)\right]|-++\rangle = |-++\rangle + |+-+\rangle$$

$$\left[e-(13)\right]\left[e+(12)\right]|-++\rangle = |-++\rangle - |++-\rangle + \cancel{|+-+\rangle} - \cancel{|+-+\rangle} = -\sqrt{2}\,|\psi_1\rangle$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \left[e-(12)\right]\left[e+(13)\right]$$

$$\left[e+(13)\right]|++-\rangle = |++-\rangle + |-++\rangle$$

$$\left[e-(12)\right]\left[e+(13)\right]|++-\rangle = |-++\rangle - |+-+\rangle = \sqrt{2}\,|\psi_2\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|-++\rangle - |+-+\rangle\right)$$

$$\left[e-(12)\right]\left[e+(13)\right]|+-+\rangle = -2\sqrt{2}\,|\psi_2\rangle$$

$$\left[e-(12)\right]\left[e+(13)\right]|-++\rangle = \sqrt{2}\,|\psi_2\rangle$$

So there are $\underline{\text{two}}$ l.c.'s that transform as $\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$,

$$\underline{|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|{+}{+}{-}\rangle - |{-}{+}{+}\rangle\right)} \qquad \equiv |\tilde{\psi}_1\rangle$$

and

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|{-}{+}{+}\rangle - |{+}{-}{+}\rangle\right).$$

Actually these are not $\underline{\text{orthonormal}}$, since $\langle\psi_2|\psi_1\rangle = -\frac{1}{2}$.

We can remove the component of $|\psi_2\rangle$ along $|\psi_1\rangle$,

$$|\tilde{\psi}_2\rangle = \left(|\psi_2\rangle - \langle\psi_1|\psi_2\rangle|\psi_1\rangle\right) / \sqrt{\text{norm}}$$

$$\underline{|\tilde{\psi}_2\rangle = \frac{1}{\sqrt{6}}\left(|{-}{+}{+}\rangle + |{+}{+}{-}\rangle - 2|{+}{-}{+}\rangle\right)}$$

This pair forms an orthonormal set that transform according to the $\underset{\sim}{2}$ rep $\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$ of S_3.

The set we used previously,

$$|\underset{\sim}{2}\rangle_1 = \frac{1}{\sqrt{3}}\left(|{+}{+}{-}\rangle + \eta|{+}{-}{+}\rangle + \eta^2|{-}{+}{+}\rangle\right) \qquad \eta = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and

$$|\underset{\sim}{2}\rangle_2 = \frac{1}{\sqrt{3}}\left(|{+}{+}{-}\rangle + \eta^2|{+}{-}{+}\rangle + \underset{\eta}{\underset{\shortparallel}{\eta^4}}|{-}{+}{+}\rangle\right)$$

are linear combinations of these,

$$|\underset{\sim}{2}\rangle_1 = \frac{1}{\sqrt{6}}(1-\eta^2)|\tilde{\psi}_1\rangle + \frac{1}{3\sqrt{2}}(1-\eta)^2|\tilde{\psi}_2\rangle$$

$$|\underset{\sim}{2}\rangle_2 = \frac{1}{\sqrt{6}}(1-\eta)|\tilde{\psi}_1\rangle + \frac{1}{3\sqrt{2}}(1-\eta^2)^2|\tilde{\psi}_2\rangle$$

So the matrices we construct using these different basis states sets will be equivalent (not =, but related by a similarity transformation).

A quick example of the matrices

recall
$$O_T \psi_\mu(\vec{x}) = \sum_\nu \mathcal{D}_{\nu\mu}(T)\, \psi_\nu(\vec{x})$$

Let's do (12), 1st on our original basis states

$$(12)\,|\underset{\sim}{2}\rangle_1 = \frac{1}{\sqrt{3}}\Big(|{+}{+}{-}\rangle + \eta\,|{-}{+}{+}\rangle + \eta^2\,|{+}{-}{+}\rangle\Big)$$

$$= |\underset{\sim}{2}\rangle_2$$

$$(12)\,|\underset{\sim}{2}\rangle_2 = \frac{1}{\sqrt{3}}\Big(|{+}{+}{-}\rangle + \eta^2\,|{-}{+}{+}\rangle + \underset{\substack{\shortparallel \\ \eta}}{\eta^4}\,|{+}{-}{+}\rangle\Big)$$

$$= |\underset{\sim}{2}\rangle_1$$

$$(12)\begin{bmatrix} |\underset{\sim}{2}\rangle_1 \\ |\underset{\sim}{2}\rangle_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathcal{D}^{T(\underset{\sim}{2})}((12))} \begin{bmatrix} |\underset{\sim}{2}\rangle_1 \\ |\underset{\sim}{2}\rangle_2 \end{bmatrix}$$

$\mathcal{D}^{T(\underset{\sim}{2})}((12))$ with the $|\underset{\sim}{2}\rangle_1, |\underset{\sim}{2}\rangle_2$ basis set.

$\boxed{= \mathcal{D}^{(\underset{\sim}{2})}((12))}$ $\quad \chi((12)) = 0$ as expected from the character table.

A bit more interesting

$$(13)\,|\underset{\sim}{2}\rangle_1 = \frac{1}{\sqrt{3}}\Big(|{-}{+}{+}\rangle + \eta\,|{+}{-}{+}\rangle + \eta^2\,|{+}{+}{-}\rangle\Big)$$

$$= \eta^*\,|\underset{\sim}{2}\rangle_2 = e^{-2\pi i/3}\,|\underset{\sim}{2}\rangle_2$$

$$(13)\,|\underset{\sim}{2}\rangle_2 = \frac{1}{\sqrt{3}}\left(|-++\rangle + \eta^2\,|+-+\rangle + \eta^4\,|++-\rangle\right)$$

$$= \eta\,|\underset{\sim}{2}\rangle_1 = e^{2\pi i/3}\,|\underset{\sim}{2}\rangle_1$$

$$(13)\begin{bmatrix}|\underset{\sim}{2}\rangle_1\\ |\underset{\sim}{2}\rangle_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & e^{-2\pi i/3}\\ e^{+2\pi i/3} & 0\end{bmatrix}}_{\mathscr{D}^T}\begin{bmatrix}|\underset{\sim}{2}\rangle_1\\ |\underset{\sim}{2}\rangle_2\end{bmatrix}$$

so $$\boxed{\mathscr{D}^{(\underset{\sim}{2})}\big((13)\big) = \begin{bmatrix}0 & e^{+2\pi i/3}\\ e^{-2\pi i/3} & 0\end{bmatrix}}$$

similarly

$$\boxed{\mathscr{D}^{(2)}\big((23)\big) = \begin{bmatrix}0 & e^{-2\pi i/3}\\ e^{+2\pi i/3} & 0\end{bmatrix}}.$$

$\Bigg\}\ \chi = 0$ ✓

$$\mathscr{D}^{(2)}\big((123)\big) = \begin{bmatrix}e^{+2\pi i/3} & 0\\ 0 & e^{-2\pi i/3}\end{bmatrix}$$

$$\mathscr{D}^{(2)}\big((132)\big) = \begin{bmatrix}e^{-2\pi i/3} & 0\\ 0 & e^{+2\pi i/3}\end{bmatrix}$$

$\Bigg\}\ \chi = -1$ ✓

$$\mathscr{D}^{(2)}(e) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Big\}\ \chi = 2$$

which completes the construction of a set of representation matrices for the $\underline{2}$ rep of S_3.

In this construction of basis states we left the problem of finding S_N irreps behind.

Just a quick reminder of how this goes.
n.b. prev. we were making states of def total spin, staying within 1 symm group, SU(2). Now we are building up IRs of a larger group.

S_1 e operates on a single object $|a\rangle$ $\square$

1 irrep of S_1

S_2 e, (12) operates on a state $|ab\rangle$ ($a \neq b$, they are object labels) distinguishable

$$\square \otimes \square = \ydiagram{2} \oplus \ydiagram{1,1}$$

2 irreps of S_2

"partition" (2) (1^2)

trivial, 1 state — alt, 1 state

$$\frac{1}{\sqrt{2}}\left(|ab\rangle + |ba\rangle\right) \qquad \frac{1}{\sqrt{2}}\left(|ab\rangle - |ba\rangle\right)$$

S_3

$$\square \otimes \square \otimes \square = \left(\ydiagram{2} \otimes \square\right) \oplus \left(\ydiagram{1,1} \otimes \square\right)$$

operates on a state $|abc\rangle$, $a \neq b \neq c$

$$= \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

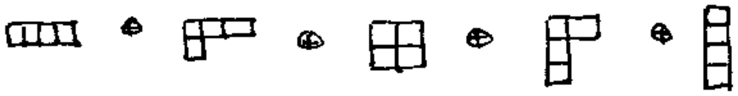
3 irreps. of S_3

"partition" (3) (2,1) already found (1^3)

trivial rep. 1 state. — the 2 mixed symm. rep. — alt. rep. 1 state

$S_4 \quad \square \otimes \square \otimes \square \otimes \square \quad =$

$|abcd\rangle$
$a \neq b \neq c \neq d$



($\oplus$ again redundant copies)

"partition" $\quad (4) \quad (3,1) \quad (2^2) \quad (2,1^2) \quad (1^4)$

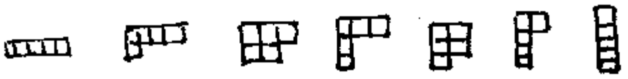
trivial rep $\qquad\qquad\qquad$ alt rep

$\underset{\sim}{1}\,, \quad \underset{\sim}{3}\,, \quad \underset{\sim}{2}\,, \quad \underset{\sim}{3}\,, \quad \underset{\sim}{1}$

5 IRs of S_4

There is a general procedure for determining the dimensionality of an S_N IR from its' Young tableaux, known as the "hook rule."

continuing...



S_5 $\quad$ 7 IRs $\qquad \underset{\sim}{1} \quad \underset{\sim}{4} \quad \underset{\sim}{5} \quad \underset{\sim}{6} \quad \underset{\sim}{5} \quad \underset{\sim}{4} \quad \underset{\sim}{1}$

S_6 $\quad$ 11 IRs $\qquad \underset{\sim}{1} \quad \underset{\sim}{5} \quad \underset{\sim}{9} \quad \underset{\sim}{10} \quad \underset{\sim}{5} \quad \underset{\sim}{16} \quad \underset{\sim}{5} \quad \underset{\sim}{10} \quad \underset{\sim}{9} \quad \underset{\sim}{5} \quad \underset{\sim}{1}$

S_7 $\quad$ 15 IRs $\qquad \ldots$

no simple rule for numbers or dims. of IRs.