Re-presentation theory [relevance of group thy. to physics]

We may represent group elements explicitly by matrices

$$\left.\begin{array}{lll} g_1 \longrightarrow \mathcal{D}(g_1) & & \text{an } N\times N \text{ matrix} \\ g_2 \rightarrow \mathcal{D}(g_2) & & \text{another} \\ \ldots & & \\ g_n \longrightarrow \mathcal{D}(g_n) & & \text{an } n^{\underline{th}} \text{ such matrix.} \end{array}\right\} \begin{array}{l} \mathcal{D}(g_1 g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2) \\ \mathcal{D}(g_1^{-1}) = [\mathcal{D}(g_1)]^{-1} \\ \mathcal{D}(e) = \underline{I} \end{array}$$

This set of matrices $\{\mathcal{D}(g_1), \ldots \mathcal{D}(g_n)\}$ is called a <u>representation</u> of the group $\mathcal{G}$. They have the same multiplication table as the group elements under matrix mult.

One rep $\rightarrow \mathcal{D}(g_i) = 1 \;\forall i$. "Trivial representation." $\underline{1}$ (1-dimensional)

$g_{m_1} \times g_{m_2} = g_{m_3} \rightarrow \underline{1} \times \underline{1} = \underline{1}$ ✓

A <u>faithful</u> representation is one in which each group element is mapped into a different matrix, so we can infer the group multiplication table from the set $\{\mathcal{D}(g)\}$. The trivial rep. is clearly not faithful.

Faithful rep = isomorphism between group $\mathcal{G}$ and matrix group $\{\mathcal{D}(g)\}$
Not faithful, = homomorphism " " " " " " " .

Representations typically arise in physics by looking at the effect of a transformation on the configuration space. Specific example of parity transformation in 1D operating on wavefunctions:

$$e\psi(x) = \psi(x) \qquad P\psi(x) = \psi(-x). \qquad P^2 = e$$

$$e^2 = e \quad eP = P \quad Pe = P$$

If we write the wavefunction as a column vector,

$$\begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix},$$

then we have

$$e \begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix} = \begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{D}(e) = I \ \checkmark} \begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix}$$

$$P \begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix} = \begin{bmatrix} \psi(-x) \\ \psi(x) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathcal{D}(P)} \begin{bmatrix} \psi(x) \\ \psi(-x) \end{bmatrix}$$

$$e^2 = e \qquad \mathcal{D}(e)\mathcal{D}(e) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{D}(e) \ \checkmark$$

$$P^2 = e \qquad \mathcal{D}(P)\mathcal{D}(P) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{D}(e) \ \checkmark$$

For a more general coordinate transformation $x' = Tx$ (e.g. 3D rotation, T_{ij} an orthog matrix) we may have a set of functions that mix among themselves under the transformation. e.g. $\psi_1(\vec{x}) = x$, $\psi_2(\vec{x}) = y$, $\psi_3(\vec{x}) = z$ only mix among themselves under a rotation.

Generally we write (for a set of n such functions)

$$\underbrace{O_T}_{\substack{\text{operation of } T \\ \text{on function } \psi_\mu}} \psi_\mu(x) = \sum_{\nu=1}^{n} \mathcal{D}^{(n)}_{\nu\mu}(T)\,\psi_\nu(x) \qquad = \psi_\mu(T^{-1}x)$$

$$O_T\psi_\mu(Tx) = \psi_\mu(x)$$

With this funny order of indices we have for the result of two successive operations

$$\mathcal{O}_{SR}\psi_\mu = \mathcal{O}_S\,\mathcal{O}_R\,\psi_\mu = \mathcal{O}_S \sum_\nu \mathcal{D}^{(n)}_{\nu\mu}(R)\,\psi_\nu$$

$$= \sum_{\nu\sigma} \mathcal{D}^{(n)}_{\nu\mu}(R)\,\mathcal{D}^{(n)}_{\sigma\nu}(S)\,\psi_\sigma$$

$$= \sum_{\nu\sigma} \left[\mathcal{D}^{(n)}_{\sigma\nu}(S)\,\mathcal{D}^{(n)}_{\nu\mu}(R)\right]\psi_\sigma$$

$$= \sum_\sigma \left[\mathcal{D}^{(n)}(S)\cdot\mathcal{D}^{(n)}(R)\right]_{\sigma\mu}\psi_\sigma$$

$$\equiv \sum_\sigma \mathcal{D}^{(n)}_{\sigma\mu}(SR)\,\psi_\sigma$$

So the rep of a combined op $\mathcal{D}^{(n)}(SR) = \mathcal{D}^{(n)}(S)\cdot\mathcal{D}^{(n)}(R)$, just the conventional matrix product

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In additional to the trivial rep $\mathcal{D}(g) = 1 \ \forall g$, two other reps are easy to write down. One is the "antisymmetric" rep,

$$\mathcal{D}(g) = \begin{cases} +1 & \text{if the order of } g \text{ is odd (smallest } m \ni g^m = e), \\ -1 & \text{" " " " " " even.} \end{cases}$$

To focus on specific e.g., recall S_3 with elements e (12) (13) (23) (123) (132)

"trivial rep." $\mathcal{D}(g) = 1 \quad \forall g$

antisymmetric rep

$$\mathcal{D}(g) = \begin{cases} 1 & e, \\ & (123), (132) \quad \text{odd order (this def } e \text{ of order 1, } \underset{m}{\text{Min}}\, e^m = e \to m=1) \\ -1 & (12), (13), (23) \quad \text{even order} \end{cases}$$

The remaining "easy" rep. to write down is the "regular representation", which can be written directly from the group multiplication table.
The definition is

$$\mathcal{D}^{(n)}_{rc}(g) = \begin{cases} 1 \text{ if } & g\, g_c = g_r \\ 0 & \text{otherwise} \end{cases} .$$

The way it works is to rep each group element as a column vector,

$$g_1 \rightarrow \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{|g_1\rangle} \qquad g_2 \rightarrow \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{|g_2\rangle} \qquad g_3 \rightarrow \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}}_{|g_3\rangle}$$

so e.g. if $g_1 g_2 = g_3$,

on 2 produces 3

$$\mathcal{D}(g_1) \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{|g_2\rangle} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}}_{|g_3\rangle} , \text{ so } \mathcal{D}(g_1) = \begin{array}{c} \\ r\downarrow \\ \\ \end{array} \begin{array}{c|ccc|} \multicolumn{4}{c}{\xrightarrow{c}} \\ & 1 & 2 & \\ 1 & & 0 & \\ 2 & & 0 & \\ 3 & & 1 & \\ & & 0 & \\ & & 0 & \\ & & \vdots & \\ \multicolumn{4}{c}{\quad\;\; |g_3\rangle} \end{array}$$

so $\mathcal{D}_{rc}(g_1) = 1$ if $g_1 g_c = g_r$ ✓

Note the dimensionality of the reg rep = order of the group.

Note we can trivially change one rep to another by a similarity transformation.
For any group element $a, b, c \in \mathcal{G}$,

$$ab = c \xrightarrow[\text{rep}]{} \mathcal{D}(a)\mathcal{D}(b) = \mathcal{D}(c)$$

Under a unitary transformation, we get a new set of matrices $\tilde{\mathcal{D}}(g) = \mathcal{U}^{\dagger}\mathcal{D}(g)\mathcal{U}$
$\mathcal{U}^{\dagger} = \mathcal{U}^{-1}$

note

$$\tilde{\mathcal{D}}(a)\tilde{\mathcal{D}}(b) = \mathcal{U}^{\dagger}\mathcal{D}(a)\underbrace{\mathcal{U}\mathcal{U}^{\dagger}}_{I}\mathcal{D}(b)\mathcal{U} = \mathcal{U}^{\dagger}\mathcal{D}(a)\mathcal{D}(b)\mathcal{U} = \mathcal{U}^{\dagger}\mathcal{D}(c)\mathcal{U} = \tilde{\mathcal{D}}(c)$$

So the new set $\{\tilde{\mathcal{D}}(g)\}$ are <u>also</u> a rep.

Since a similarity transform is just a change of basis, or coord sys reps related by one are essentially identical and are called "equivalent".
Hence the matrices alone are not a very useful way of describing a representation. More useful way is to quote the "characters", which are the <u>traces</u> of the matrices,

$$\chi(g) \equiv \mathrm{Tr}\left[\mathcal{D}(g)\right].$$

This is invariant under a similarity transformation,

$$\chi(g) = \mathrm{Tr}\left[\mathcal{D}(g)\right],$$

$$\tilde{\chi}(g) = \mathrm{Tr}\left[\tilde{\mathcal{D}}(g)\right] = \mathrm{Tr}\left[\mathcal{S}^{\dagger}\mathcal{D}(g)\mathcal{S}\right] = \mathrm{Tr}\left[\mathcal{D}(g)\underbrace{\mathcal{S}\mathcal{S}^{\dagger}}_{I}\right] = \chi(g).$$

Note the $\chi(g)$ are <u>class</u> functions, if $g_1 = g g_2 g^{-1}$, then $\chi(g_1) = \chi(g_2)$.
A "character table" has one entry for each class.

Reducibility and Irreducible representations.

Given a set of matrices $\{\mathcal{D}(g)\}$ that represent a group $\mathcal{G}$, it may be possible to block-diagonalize them by a similarity transformation,

$$\mathcal{S}^{\dagger}\mathcal{D}(g)\mathcal{S} = \left[\begin{array}{c|c|c|c} \mathcal{D}_1(g) & & & \\ \hline & \boxed{\mathcal{D}_2(g)} & & \bigcirc \\ & & \boxed{\mathcal{D}_3(g)} & \\ \hline & \bigcirc & & \mathcal{D}_4(g) \end{array}\right] \quad \forall\, g, \text{ i.e.}$$

the rep $\mathcal{D}(g)$ is really the sum of smaller matrices which cannot be reduced any further in size.
We write this as

$$\mathcal{D}(g) = \mathcal{D}_1(g) \oplus \mathcal{D}_2(g) \oplus \mathcal{D}_3(g) \oplus \mathcal{D}_4(g) \quad \text{and say}$$

$\mathcal{D}(g)$ is the "direct sum" of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$. (Any e.g. $\mathcal{D}_1$ could appear $>$once or not at all.)

A rep that can be block-diagonalized is referred to as "reducible", and the reps like $\mathcal{D}_1 \ldots \mathcal{D}_4$ that cannot be reduced to smaller reps are known as "irreducible" reps.

A general rep $\mathcal{D}(g)$ can be decomposed into irreducible representations, and for a finite group the number of irreps is also finite.

Fund results (not proven) : 1) # of irreducible reps of $\mathcal{G}$ = # of classes κ

2) the dimensionalities $\{d_i\}$ of the matrices in the i^{th} irred rep satisfy

$$\sum_{i=1}^{\kappa} d_i^2 = n,$$

where n is the order of the group.

Specific example of S_3 : we learn from group mult table and looking for equivalent elements that S_3 has $\kappa = 3$ classes, therefore it has 3 irreducible reps. Since it has $S_n \to n! = 6$ elements,

$$\sum_{i=1}^{\kappa=3} d_i^2 = n = 6.$$

The only 3 pos. integers $\ni$ $d_1^2 + d_2^2 + d_3^2 = 6$ are 1, 1, 2, so S_3 has two 1d irred reps (trivial and antisymmetric reps), and an irred rep of 2D matrices. (These 2D matrices are given explicitly by M&W on p.433, table 16-1.)

Orthogonality relations of representations & characters.

Also derived from Schur's lemmas, see e.g. Hamermesh Ch 3.

Suppose we have two irreducible reps of a finite group $\mathcal{G}$ in terms of unitary matrices ($\mathcal{D}^{+} = \mathcal{D}^{-1}$). (Can easily convert a nonunitary rep into a unitary one by a similarity trans: Wigner gives the trans explicitly Ch 9 Theorem I).

for irreducible representations

If the order of the group is n, the representation matrices satisfy

$$\boxed{\sum_{g} \mathcal{D}^{(r_1)}_{i_1 j_1}(g)\, \mathcal{D}^{(r_2)}_{i_2 j_2}(g)^{*} = \underbrace{\frac{n}{d_1}}_{\substack{\text{dimension}\\ \text{of } r_1\\ \text{matrices}}} \underbrace{\delta_{r_1 r_2}}_{\substack{\text{must}\\ \text{be}\\ \text{same}\\ \text{rep}}} \delta_{i_1 i_2}\delta_{j_1 j_2}}$$

Orthog theorem for group representation matrices.

Note this implies that the total # of irred reps of a finite group is itself finite: Can think of the $\{\mathcal{D}^{(r_1)}_{ij_1}\}$ as a set of d_1^2 vectors

$$\mathrm{vec}\rangle = \left\{ \underbrace{\mathcal{D}^{(r_1)}_{11}(e)}_{\text{a number}}, \underbrace{\mathcal{D}^{(r_1)}_{11}(g_1)}_{\text{a number}}, \mathcal{D}^{(r_1)}_{11}(g_2), \ldots, \mathcal{D}^{(r_1)}_{11}(g_{n-1}) \right\} = \text{a } 1^{\underline{st}} \text{ n-component vector}$$

$$\widetilde{\mathrm{vec}}_{d_1^2}\rangle = \left\{ \mathcal{D}^{(r_1)}_{d_1 d_1}(e), \mathcal{D}^{(r_1)}_{d_1 d_1}(g_1), \ldots \qquad , \mathcal{D}^{(r_1)}_{d_1 d_1}(g_{n-1}) \right\} = \text{a } (d_1^2)^{\underline{th}} \text{ n-component vector}$$

& sim for r_2

And all these vectors are orthogonal in this n-dimensional space.
Since you can have <u>at</u> <u>most</u> n orthog vectors in an n-dim space, we have

$$\sum_r d_r^2 \leq n$$

So even if all irreps were 1-dim you could have at most n of them.
(This is the case for Abelian groups : all unit. irreps. of an Abelian group are 1D.)

In fact this is an equality, (Proof straightforward in Homework)
(Uses decom of reg. rep into irreps)

$$\boxed{\sum_r d_r^2 = n}$$

Σ over all irred reps

their dim^2

order of the group

This can sometimes be solved uniquely to specify the dims of irreps of a group.

We are really more interested in the characters of an irrep, since they are invariant under similarity transformation.
Taking the trace $\delta_{i_1j_1}\delta_{i_2j_2}$ of our representation matrix orthog relation, we have

$$\sum_g \chi^{(r_1)}(g)\,\chi^{(r_2)}(g)^* = \frac{n}{d_1}\delta_{r_1r_2}\underbrace{\delta_{i_1i_2}\delta_{i_1i_2}}_{\delta_{i_1i_1}=d_1} = n\,\delta_{r_1r_2}$$

or, since the characters $\chi(g)$ are the same for each class,

$$\boxed{\sum_k p_k\,\chi^{(r_1)}(C_k)\,\chi^{(r_2)}(C_k)^* = n\,\delta_{r_1r_2}}$$

We can sometimes use these results to set up the "character table" for a group, which lists the character $\chi^{(r_i)}(C_k)$ of each class C_k in each irrep r_i. Since the # classes = # irreps, it is a square table.

For the example of S_3, we already know we have the identity or trivial rep

$$\mathcal{D}(g) = 1 \quad \forall\, g \quad \rightarrow \quad \chi(C_k) = 1 \quad \forall\, C_k$$

& antisymm rep

$$\mathcal{D}(g) = 1 \quad g = e, (123) \longrightarrow \chi(C_k) = 1 \quad k = 1, 2$$

$$\mathcal{D}(g) = -1 \quad g = (12) \longrightarrow \chi(C_k) = -1 \quad k = 3$$

elems in class

rep

e.g. ~~class~~	p_k	class	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$
e	1	C_1	1	1	2
(123)	2	C_2	1	1	$\alpha = -1$
(12)	3	C_3	1	-1	$\beta = 0$

& we know there is a 2D rep in addition, since

$$\sum_{r=1}^{3} d_r^2 = n = 6$$

$$1^2 + 1^2 + d_3^2 = 6$$

$$\rightarrow d_3 = 2$$

For the $\underset{\sim}{2}$ rep, $\mathcal{D}(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\chi(e) = 2$

The remaining elements in the character table can be found from the orthogonality relation for characters of irreps,

$$\sum_{k=1}^{3} p_k\, \chi^{(r_1)}(C_k)\, \chi^{(r_2)}(C_k)^* = n\, \delta_{r_1 r_2}$$

$$\left.\begin{aligned} r_1 = (1),\ r_2 = (3) &\rightarrow 1\cdot 1\cdot 2 + 2\cdot 1\cdot \alpha^* + 3\cdot 1\cdot \beta^* = 0 \\ r_1 = (2),\ r_2 = (3) &\rightarrow 1\cdot 1\cdot 2 + 2\cdot 1\cdot \alpha^* + 3\cdot(-1)\cdot \beta^* = 0 \end{aligned}\right\} \quad \Sigma \,\&\, \text{diff},\ \alpha^* = -1, \quad \beta^* = 0$$

n.b. $r_1 = (1)$ $r_2 = (2) \rightarrow 1\cdot 1\cdot 1 + 2\cdot 1\cdot 1 + 3\cdot 1\cdot(-1) = 0$ ✓ n.b. $r_1 = r_2 = (3)$ $\quad 1\cdot 2^2 + 2\cdot(-1)^2 + 3\cdot 0^2 = 6$ ✓ nontrivial check

Actually can think of $\frac{1}{\sqrt{n}} \left\{ \sqrt{p_k}\, \chi^{(r)}(c_k) \right\}$ $k = 1, 2, \ldots N_{\text{classes}} \equiv \mathbb{K}$

as one of a set of $\mathbb{K}$ orthonormal vectors in a $\mathbb{K}$-dimensional space.

e.g. $1^{\underline{st}} = \frac{1}{\sqrt{6}} \left(1, \sqrt{2}, \sqrt{3} \right)$

$2^{\underline{nd}} = \frac{1}{\sqrt{6}} \left(1, \sqrt{2}, -\sqrt{3} \right)$

$3^{\underline{rd}} = \frac{1}{\sqrt{6}} \left(2, -\sqrt{2}, 0 \right)$

& you can check that these are an orthonormal set.