Group theory in quantum mechanics

Refs Mathews & Walker Ch 16
Hamermesh
Wigner

What is a group? $\mathcal{G}$

= A set $\{g_1, g_2, \ldots, g_n\}$ of n elements (may be a continuous infinity) that satisfy four axioms:

1. A multiplication "product" is defined within the group that maps any two elements onto a third, $g_{m_1} \times g_{m_2} = g_{m_3}$ $\forall\ m_1, m_2$ (hence group is closed under multiplication.)

2. Multiplication is associative,

$$g_1 \times (g_2 \times g_3) = (g_1 \times g_2) \times g_3 .$$

(But non commutative in general, $g_1 \times g_2 \neq g_2 \times g_1$. Groups which are commutative are called Abelian groups.)

3. There is an identity element e in the set,

$$g_1 \times e = e \times g_1 = g_1 \qquad \& \ldots .$$

4. There is an inverse g_1^{-1} for every element g_1,

$$g_1 \times g_1^{-1} = g_1^{-1} \times g_1 = e .$$

If the number of elements is finite we refer to this as a "finite group".
elements = "order" of the group.

Examples of groups :

parity operation in 1D.

$$\mathcal{P} f(x) = f(-x) \qquad \mathcal{P}^2 f(x) = f(x) = e f(x).$$

(this is a transformation group, since there is an underlying $f(\vec{x})$ we are operating on.)

Group has 2 elements, P and e.

Group multiplication table

$g_1 \downarrow \; g_2 \rightarrow$	e	P
e	e	P
P	P	e

$g_1 \times g_2 =$ entry

closure $\mathcal{P} \times e = \mathcal{P}$ ✓ $\quad P \times P = P^2 = e$ ✓ $\quad e \times e = e$ ✓

assoc ✓

identity ✓ (do not invert coords) = e $\qquad$ (invert coords) = P

inverse $\mathcal{P}^2 = e$, so $\mathcal{P}^{-1} = P$ ✓

Note this is trivially an Abelian group,
$g_1 \times g_2 = g_2 \times g_1$ $\forall$ elements (P, e only)

S_n contains all finite groups:

Cayley's theorem:

Every group $\mathcal{G}$ of order n is isomorphic with a subgroup of S_n.

Symmetric group $\boxed{S_n}$.

This is the permutation group on n objects. A general permutation can be written as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix},$$

this operation takes object 1 into p_1, 2 into $p_2, \ldots,$ n into p_n. The order of S_n is clearly $n!$

If we just switch 2 objects, e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix}$ this is just called a "transposition" and can be written simply as (12) (the "cycle"): $1 \to 2$ and $2 \to 1$. More complicated permutations, e.g. S_3 with $3! = 6$ elements, can also be written as cycles:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = (123) \qquad 1\to 2,\ 2\to 3,\ 3\to 1$$

$$= (13)(12) \qquad \text{① } 1\to 2,\ 2\to 1 \quad \text{② } (2\to 1)\to 3,\ 3\to 1 \quad \Bigg\} \begin{matrix} 1\to 2 \\ 2\to 3 \\ 3 \to 1\end{matrix}$$

$$(13)(12)|abc\rangle = (13)|bac\rangle = |cab\rangle = (123)|abc\rangle$$

$\boxed{\text{Similarly any perm can be written as the product of transpositions.}}$

Can think of S_n operating on $|a\, b\, c\, d \ldots L_n\rangle$, e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} |\overset{1^{st}}{a}\, \overset{2^{nd}}{b}\, \overset{3^{rd}}{c}\rangle = |\overset{1^{st}}{c}\, \overset{2^{nd}}{a}\, \overset{3^{rd}}{b}\rangle$$

S_n is non-Abelian for $n \geq 3$.

n.b. The order of a group element is the power you must raise it to to get e. e.g. $\begin{cases} (12) & \text{elem of order 2} \\ (123) & \text{elem of order 3} \end{cases}$

Subgroups

A group $\mathcal{G}$ is the set $\{e, g_1, g_2, \ldots, g_n\}$ of $n+1$ elements closed under mult, with identity and inverse.

If we can find a subset of elements $\{e, \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m\}$ that also closes under mult, has e + inverse, that's known as a subgroup $\tilde{\mathcal{G}}$ of $\mathcal{G}$,

$$\tilde{\mathcal{G}} \subset \mathcal{G}$$

example: Symmetric group S_3 contains 6 elements $\begin{pmatrix}123\\123\end{pmatrix}$ $\begin{pmatrix}123\\231\end{pmatrix}$ $\begin{pmatrix}123\\312\end{pmatrix}$

$\begin{pmatrix}123\\132\end{pmatrix}$ $\begin{pmatrix}123\\321\end{pmatrix}$ $\begin{pmatrix}123\\213\end{pmatrix}$

① The subset $\begin{pmatrix}123\\123\end{pmatrix}$ $\begin{pmatrix}123\\231\end{pmatrix}$ $\begin{pmatrix}123\\312\end{pmatrix}$ that maintain the order of $|abc\rangle$ (as if they are on a cyclic ring) is known as the cyclic group C_3.

The cyclic group C_n clearly consists of the powers of a single element, here

(123), $(123)(123) = (132)$, $(123)(123)(123) = e$.

could call $(123) \equiv a$, then $C_3 = \{e, a, a^2\}$

(C_n in general). Clearly it is Abelian & is order n.

② Can divide S_n into elements that contain an even or odd number of transpositions.

$\begin{pmatrix}123\\123\end{pmatrix} = e$ even	$\begin{pmatrix}123\\231\end{pmatrix} = (123) = (13)(12)$ even	$\begin{pmatrix}123\\312\end{pmatrix} = (132) = (12)(13)$ even
$\begin{pmatrix}123\\132\end{pmatrix} = (23)$ odd	$\begin{pmatrix}123\\321\end{pmatrix} = (13)$ odd	$\begin{pmatrix}123\\213\end{pmatrix} = (12)$ odd

Clearly the product of even transpositions is also an even transposition.

The group of <u>even</u> transpositions is called the "alternating group" A_n, and is of order $n!/2$.

What size (order) subgroups are allowed?

> Lagrange's theorem: The order of a subgroup of a finite group is a divisor of the order of the group.
> (Not much help for S_n!)

Can however use this to find all possible group structures of given order.
e.g. a group of order 6 can have subgroups of order 1, 2, 3 or 6.

For cyclic group C_6 with elements $a, a^2, a^3, a^4, a^5, a^6 = e$,
there are

$$e \; ; \quad \underbrace{(e, a^3)}_{C_2,\ \text{isomorphic to } S_2} \; ; \quad \underbrace{(e, a^2, a^4)}_{C_3}$$

<u>Classes.</u> Two group elements g_1 and g_2 belong to the same "class" if for some element g

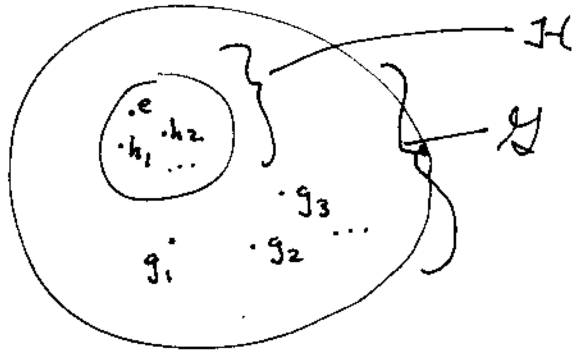
$$g g_1 g^{-1} = g_2 \qquad \Longrightarrow \ g_1 \text{ and } g_2 \text{ are "equivalent"}$$

actually called "conjugate".

The complete set of group elements which are equivalent to each other (you may change g) form a "class".

Note that the identity is in a class by itself. In an Abelian group, <u>every</u> element is a class by itself.

Note also to belong to the same class, two group elements must be of the same order.



<u>Invariant</u> subgroups

We previously discussed the idea of a subgroup $\mathcal{H}$ of a group $\mathcal{G}$.

A subgroup $\mathcal{H}$ that contains all conjugate elements, i.e. contains all members of a class if it contains any, is known as an "invariant subgroup":

$$\text{If} \quad \underbrace{g\mathcal{H}g^{-1}}_{\text{the set}} = \mathcal{H} \quad \text{for} \quad \forall g \in \mathcal{G},$$

then $\mathcal{H}$ is an "invariant subgroup".

If $\mathcal{G}$ contains <u>no</u> invariant subgroups it is "simple".

" $\mathcal{G}$ " " " Abelian " " " "semisimple".

(Elem. part. phys. groups e.g. SU(2), SU(3), SO(3) are "simple Lie groups".)

Aside: <u>Cosets</u>

$\mathcal{G}$: $(e, h_1, h_2, \ldots, h_{m-1}; g_1, g_2, \ldots, g_{n-m-1})$ order n

For a group $\mathcal{G}$ and subgroup $\mathcal{H}$: $(e, h_1, h_2, \ldots, h_{m-1})$, order m

If g_1 is an element of $\mathcal{G}$ which is not in $\mathcal{H}$,
the set $g_1\mathcal{H} = (g_1e, g_1h_1, g_1h_2, \ldots, g_1h_{m-1})$ is called the "left coset" of g_1.

Note 1) all elements of $g_1\mathcal{H}$ are distinct. (Mult by g_1^{-1} to see.) $g_1h_i = g_1h_j \to h_i = h_j$

2) no element of $g_1\mathcal{H}$ is contained in $\mathcal{H}$. if $g_1h_i = h_j \to g_1 = h_jh_i^{-1} \in \mathcal{H}$, contradiction

So $\mathcal{H}$ and $g_1\mathcal{H}$ are disjoint sets of $2m$ elements of $\mathcal{G}$.
Continue: is there any other element g_2 not contained in $g_1\mathcal{H}$? If so, form <u>its</u> left coset $g_2\mathcal{H}$, also distinct from $g_1\mathcal{H}$. If $g_2h_i = g_1h_j$, $g_2 = g_1h_jh_i^{-1} \in g_1\mathcal{H}$, violates init def $g_1\mathcal{H} \not\ni g_2$.

Continue until $\mathcal{G}$ (the entire original group) has been generated by $\overbrace{\underbrace{\underbrace{\mathcal{H}}_{m \text{ elements}}, \underbrace{g_1\mathcal{H}}_{m \text{ elements}}, g_2\mathcal{H} \ldots g_{k-1}\mathcal{H}}_{km \text{ elements } = \mathcal{G},\ n \text{ elements}}}^{\text{Total of } k \text{ cosets.}} = \mathcal{G}$.

$\therefore\ m = n/k$ or m is a divisor of n. We have just proven Lagrange's theorem, order of subgroup is a divisor of the order of a group.

n.b. Could use "right cosets" $\mathcal{H}g$ of course.

Invariant subgroups are interesting because "coset multiplication" then has group properties.

Two cosets $g_1\mathcal{H}$, $g_2\mathcal{H}$ — $\mathcal{H}$ = the set of elements in $\mathcal{H}$, $(e, h_1, h_2, \ldots, h_{m-1})$; g_1 & g_2 in $\mathcal{G}$ but not $\mathcal{H}$.

Suppose $g_1 g_2 = g_3$.

$$(g_1\mathcal{H})(g_2\mathcal{H}) = g_1\underbrace{(\mathcal{H}g_2)}_{= g_2\mathcal{H}}\mathcal{H} = g_1 g_2 \underbrace{\mathcal{H}\mathcal{H}}_{\mathcal{H}} = g_3\mathcal{H} = \text{another coset of } \mathcal{H}.$$

$= g_2\mathcal{H}$ if $\mathcal{H}$ is an <u>invariant</u> subgroup, $g\mathcal{H}g^{-1} = \mathcal{H}\ \forall\ g \subset \mathcal{G}$

So the product of two cosets is a coset.

Also

$$\underbrace{(g_1^{-1}\mathcal{H})}_{\text{inverse element}}(g_1\mathcal{H}) = g_1^{-1}g_1\mathcal{H}^2 = \underbrace{\mathcal{H}}_{\text{identity element}}$$

$\therefore$ the set of cosets of $\mathcal{H}$ forms a group, (under ordinary group multiplication as defined for $\mathcal{G}$ and $\mathcal{H}$) with $k = \frac{n}{m}$ elements. (Each element is a set.)

This is called the "factor group" $\mathcal{G}/\mathcal{H}$.

Direct products of groups.

A group $\mathcal{G}$ is the direct product of subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\underbrace{\mathcal{G}}_{\text{order } n} = \underbrace{\mathcal{H}_1}_{\text{order } m_1} \otimes \underbrace{\mathcal{H}_2}_{\text{order } m_2}, \qquad n = m_1 \cdot m_2$$

if 1) the element of $\mathcal{H}_1$ and $\mathcal{H}_2$ commute, and
2) the decomp of any element $g \in \mathcal{G}$ in element of $\mathcal{H}_1$ and $\mathcal{H}_2$, $g = h_1 h_2$, is unique.

Might write it as $g = (h_1, h_2)$.

It follows that $\mathcal{H}_1$ & $\mathcal{H}_2$ are invariant subgroups.

perm 

$\{e, a, a^2, a^3, a^4, a^5\}$ $\quad a^6 = e$

e.g. from Hamermesh: Cyclic group $C_6 = \{e, a, a^2, a^3, a^4, a^5\}$ can be written as the direct products of subgroups

$$A = \{e, a^2, a^4\} \qquad B = \{e, a^3\}$$

$$C_6 = A \otimes B$$

$$\begin{aligned} e &= e \cdot e \\ a &= a^4 \cdot a^3 \\ a^2 &= a^2 \cdot e \\ a^3 &= e \cdot a^3 \\ a^4 &= a^4 \cdot e \\ a^5 &= a^2 \cdot a^3 \end{aligned}$$

More frequently the subgroups operate on completely different spaces, e.g. isospin and spin.