Summary of QM II Topics (pre-final)

final: open book/notes
2 probs d
1 T/F / simple questions prob.

12:30 - 2:30
Monday 5 May
Rm 306

Scattering

Transitions

Symmetries

Path $\int$s

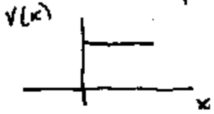
Finite Temp

Identical Particles

Scattering

1D potential scattering

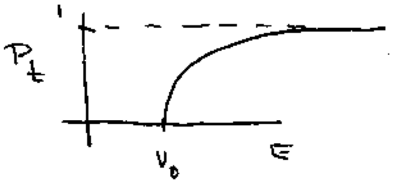
V step incident, refl & trans



$e^{ikx} + re^{-ikx}$ $te^{-x/a}$ $\left(V > \frac{\hbar^2 k^2}{2m}\right)$

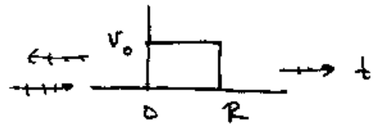
ψ, ψ' continuity b.c.

conservation of flux $\frac{p}{m} \cdot |\text{coeff}|^2$



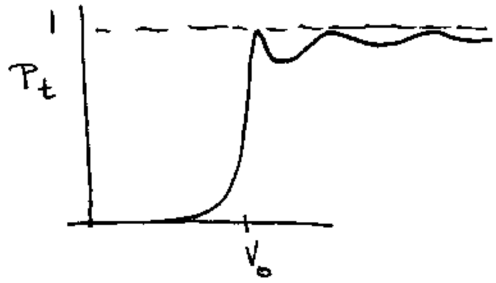
reflected wave phase shift δ

finite barrier



$e^{ikx} + re^{-ikx}$ damped or osc te^{ikx}

"tunnel effect" $E > V_0$

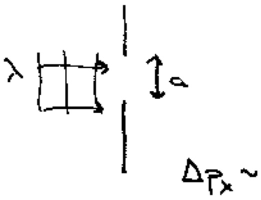


behaviour & why

trans & refl phase shifts

3D potential scat

scat. due to unc. princ.



$\Delta p_x \sim \frac{h}{a}$ → $\theta \sim \frac{\lambda}{a}$

Born series $\psi = \underbrace{\psi_{inc}}_{e^{ikz}} + \underbrace{\psi_{scat}}_{f(\Omega)\frac{e^{ikr}}{r}}$

outside V

power series in V

1^{st} Born
$$\psi^{(1)}(\vec{x}) = \underbrace{\left[E + \frac{\hbar^2}{2m}\vec{\nabla}^2\right]^{-1}} V(\vec{x})\,\psi^{(0)}(\vec{x})$$

$$K(\vec{x}-\vec{x}') = -\frac{m}{2\pi\hbar^2}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad \text{G.F. for free TISE}$$

→ Born series for wfn

$$\psi^{(1)}(\vec{x}) = \int d^3x'\, K(\vec{x}-\vec{x}')\, V(\vec{x}')\, \psi^{(0)}(\vec{x}') \quad \text{etc}$$

asymptotically recover $f(\Omega)$ in outgoing wave

$$f(\hat{\Omega}) = -\frac{m}{2\pi\hbar^2}\int d^3x'\, V(\vec{x}')\, e^{i\vec{q}\cdot\vec{x}'} \qquad \vec{q}^{\,2} = 4k^2\sin^2(\theta/2)$$

$$\frac{d\sigma}{d\Omega} = |f(\hat{\Omega})|^2$$

↳ for $V(r)$ only

$$\sigma = \int |f(\hat{\Omega})|^2 d\Omega \qquad f(q) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r\,dr\, V(r)\sin(qr)$$

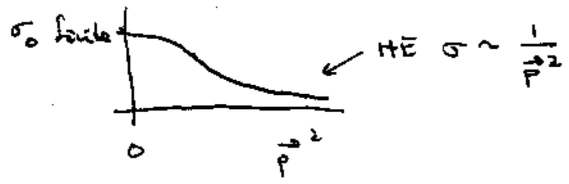
$$q = 2k\sin(\theta/2)$$

e.g. Coulomb $\quad V = -\frac{e^2}{r}$

$$\frac{d\sigma}{d\Omega} = \frac{m^2e^4}{4\vec{p}^{\,4}}\frac{1}{\sin^4(\theta/2)} \qquad \text{Rutherford cross section}$$

↳ σ divergent

also did screened Coulomb



PWA – very useful formalism, esp. for low-E scat

Expand the outgoing wave's $f(\theta)$ in Legendre functions. ($|lm\rangle$ basis scat. thy.)

$$f = \sum_l f_l P_l(\mu)$$

$\hookrightarrow$ for an incident e^{ikz}, this must be of the form

$$f_l = \frac{1}{k}(2l+1)\, e^{i\delta_l} \sin\delta_l$$ "l^{th} partial wave amplitude"

This gives a nice simple formula for

$$\sigma = \sum_{l=0}^{\infty} \underbrace{\frac{4\pi}{k^2}(2l+1)\sin^2\delta_l}_{\sigma_l \,\equiv\, l^{th} \text{ partial wave cross section}}$$

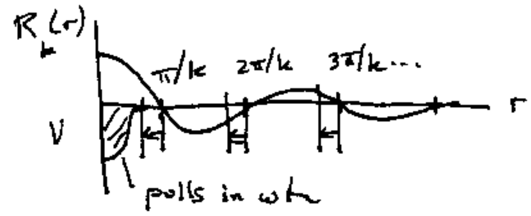
$$\sigma_l \le \frac{4\pi}{k^2}(2l+1)$$ "unitarity bound" (cons. of flux of $\psi_i = e^{ikz}$).

near threshold usually $l=0$ dominates,
so expect

$$\lim_{k\to 0} \sigma_{tot} < \frac{4\pi}{k^2}$$

Phase shifts $\{\delta_l\}$ (equiv. param. of scat amps)

e.g.
$l=0$



shift in zero is $-\dfrac{\delta_{l=0}}{k}$

attractive V, $\delta_l > 0$
repulsive V, $\delta_l < 0$ (weak V)

examples: hard sphere

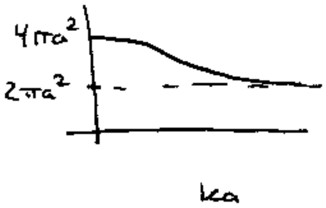
$$\delta_\ell = \tan^{-1}\left\{ \frac{j_\ell(kR)}{\eta_\ell(kR)} \right\} \qquad \delta_0 = -kR$$

generally

$$\delta_\ell \propto k^{2\ell+1} \quad \text{near threshold}$$

$$\sigma_\ell \propto k^{4\ell} \quad \text{"}$$

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)^2 \frac{1}{1+\left[\frac{\eta_\ell(kR)}{j_\ell(kR)}\right]^2}$$



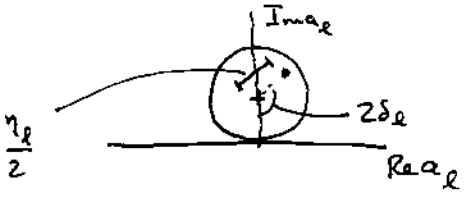
(complicated sum of many σ_ℓs)

Ramsauer–Townsend effect: if $\delta_{\ell=0} \approx \pi$, can get 'accidental' small cross section.
$E_{e^-} \approx 0.7$ eV e^- on rare-gas atoms. (idea due to Bohr)

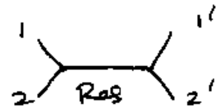
<u>Argand plot</u>

$$a_\ell = \frac{1}{2i}\left(\eta_\ell e^{2i\delta_\ell} - 1\right) \qquad \left[f(\Omega) = \frac{1}{k}\sum_\ell (2\ell+1) a_\ell P_\ell(\mu) \right]$$

↳ "inelasticity" $0 \le \eta_\ell \le 1$
(=1 for potential scat.)



resonances, as in

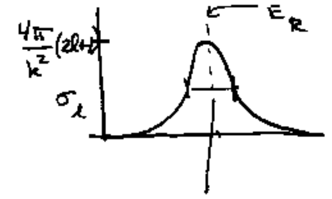


$$a_\ell \propto \frac{1}{E_R - E - i\Gamma_R/2}$$

do a "resonance loop" in the Argand plot this a_ℓ gives

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1) \cdot \frac{\Gamma^2/4}{(E-E_R)^2+\Gamma^2/4}$$

"Breit-Wigner cross section"



FWHM $= \Gamma$, $\tau = \hbar/\Gamma$

(neglecting the $\frac{1}{k^2}$ overall)

Determination of $\{\delta_\ell\}$

Subst in integral form of Schrö eqn:

$$\sin\delta_\ell = -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, R_{k,\ell}(r)\, j_\ell(kr)\, r^2\, dr \qquad \underline{\text{exact}}$$

$R_{k,\ell}$: radial wfn [$j_\ell(kr)$ free particle]

Born approx ($R \to j_\ell$, $\sin\delta_\ell \to \delta_\ell$, weak V)

$$\delta_\ell^{(\text{Born})} = -\frac{2mk}{\hbar^2}\int_0^\infty V(r)\, j_\ell(kr)^2\, r^2\, dr$$

near threshold

$$\delta_0^{(\text{Born})} \approx -\frac{mk}{2\pi\hbar^2}\int V(r)\, d^3x \quad ; \quad \delta_\ell^{(\text{Born})} \approx -\frac{2m}{\hbar^2}\, 2^{2\ell}\left[\frac{\ell!}{(2\ell+1)!}\right]^2 k^{2\ell+1}\int_0^\infty V(r)\, r^{2\ell+2}\, dr$$

$$\sigma_{tot} \approx \sigma_0 = \frac{m^2}{\pi\hbar^4}\left|V(\vec{q}=0)\right|^2 = \frac{m^2}{\pi\hbar^4}\left|\int V(r)\, d^3x\right|^2$$

recall 1st Born approx

$$\frac{d\sigma}{d\Omega} = \left|-\frac{m}{2\pi\hbar^2}V(\vec{q})\right|^2 \longrightarrow \sigma \approx 4\pi \cdot \frac{m^2}{4\pi^2\hbar^4}\cdot\left|V(q=0)\right|^2$$

Optical theorem

$$f = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\mu)$$

$$\downarrow$$

$\mathrm{Im} f$ $\qquad\qquad \sin\delta_\ell$

compare w/ σ_{tot} formula, $\qquad \underbrace{\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\, f(0)}$

(cons. of probability flux)

$\mathcal{S}$-matrix (Heisenberg 1942)

$$|i\rangle = \begin{bmatrix} c_R^i \\ c_L^i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |f\rangle = \begin{bmatrix} c_R^f \\ c_L^f \end{bmatrix} = \begin{bmatrix} t \\ r \end{bmatrix}$$

$$|f\rangle = \mathcal{S}|i\rangle$$

↳ a 2x2 complex matrix "scattering matrix"

this has all of physics (all possible expts)

$\underbrace{\mathcal{S}^\dagger = \mathcal{S}^{-1}}$ from cons. of prob.

↳ this relates the δ_r & δ_t of 1D scat.

Time dep Schrö eqn & transitions

$$\psi(\vec{x},t) = \underbrace{e^{-\frac{i}{\hbar}\mathcal{H}_{op} t}}_{U(t,0)} \psi(\vec{x},0)$$

pictures for time evolution of matrix elements

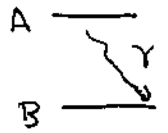
Schrödinger $|\psi\rangle = U|\psi(0)\rangle$, $\mathcal{O}$ fixed

Heisenberg $\mathcal{O}(t) = U^\dagger \mathcal{O}(0) U$, $|\psi\rangle$ fixed

time dep. pert. thy. transient examples

Fermi's golden rule

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar} |\langle f|H_I|i\rangle|^2 \rho(E_f)$$

radiative transitions 

explicit examples of Hydrogenic transitions. E1 decay rate formula.

Polarization sum (vs NIST)

Symmetry in QM

Defn of groups, subgroups, invar. subgroups

S_n examples

representations $\{\mathcal{D}(g)\}$ reducible & irreducible
triv., alt., regular

characters & char. tables.
decomposition of reps using

Schur's lemma (physics context)

 $H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$ example - states in S_3 irreps.

basis states & Young tableaux. Young symmetrizers.

Explicit group matrices. (S_3)

<u>Continuous groups & Lie groups.</u>

definitions $U = e^{i \sum_a \theta^a J^a}$

↳ generators

Lie algebra $[J^a, J^b] = \sum_c i f^{abc} J^c$

↳ structure coeffs.

Jacobi identity.

<u>All</u> Lie groups. Rank.

SU(n). Origin of SU(n) as a symm of $\mathcal{L}$ in the standard model.

Irreps of SU(2) & SU(3), lower representations & Young tableaux,

SU(3) & quark model states. 3, $\bar{3}$, $\underbrace{\underset{\sim}{8}, \underset{\sim}{10}}_{\text{baryons.}}$

Casimir ops & diagonal generators

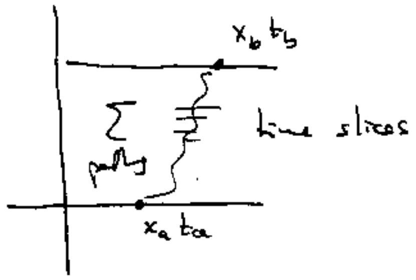
I^2 & I_z — SU(2)

I_z & Y — SU(3)

Gell-Mann matrices $\{\lambda^a\}$

Path $\int$s.

Equivalent descr of QM



$$A(x_a t_a \to x_b t_b) = \sum_{paths} e^{iS/\hbar}$$

time slicing, norm of $\int dx_k$

differential ψ evolution $\to$ recover Schrö eqn.

free theory, can do path $\int$ $\to$ kernel for free TDSE

$$K = \sqrt{\frac{m}{2\pi i \hbar t}}\, e^{\frac{imx^2}{2\hbar t}}$$

eigenmode expansion of kernels (can recover all wfns from them)

$e^{iS/\hbar}$ & the origin of non-commuting ops.

$T > 0$ QM

$$p(E) = e^{-\beta E} \qquad \beta \to \frac{1}{k_B T}$$ Boltzmann dist.

thermal averages

$$\langle O \rangle = \frac{1}{Z} \sum_n \langle \psi_n | O | \psi_n \rangle e^{-\beta E_n} = \frac{1}{Z} \mathrm{Tr} \{ O \rho \}$$

$$\rho = e^{-\beta H} = \text{"density matrix"} \qquad Z = \sum_n e^{-\beta E_n} = \mathrm{Tr}\{ e^{-\beta H} \} = \mathrm{Tr}\{\rho\}$$

= "partition function"

e.g.s of spin systems

$$E(T) = -\frac{\partial}{\partial \beta} \ln Z$$

$$c(T) = k_B \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z$$

low T properties $\propto e^{-\beta E_{gap}}$

$$\chi(T) = \frac{\beta}{Z} \sum_n \langle \psi_n | M_z^2 | \psi_n \rangle e^{-\beta E_n}$$

$Z(\beta)$ by inspection.

High temperature series from $\langle O \rangle = \dfrac{\mathrm{Tr}\{O\rho\}}{\mathrm{Tr}\{\rho\}}$ & expand $\rho = e^{-\beta H}$

Identical particles

Distinguishable vs indist. states.

Avoiding double counting by state symmetrization.

Spin – statistics theorem (from causality) (Pauli 1940)

fermions → anticommutators, hence exclusion princip.

symm/antisymm of wfns. as equivalent to symm/antisymm of states.

>1 d.o.f. : Ψ_{tot} is $\underset{\text{bose / fermi}}{S / A}$, but individual d.o.f. wfns. need not be.