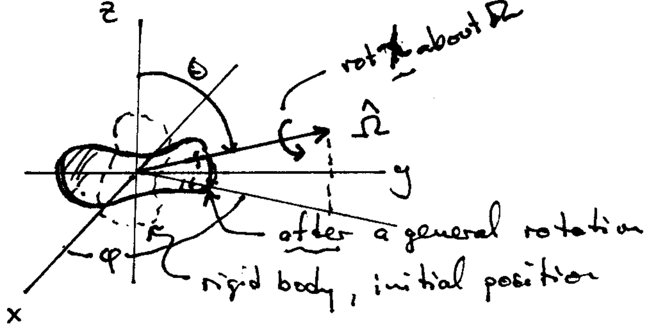
pp. 1-6

Rotations & Rot. matrices in QM (from Rose, Elem. Thy. of Ang. Mom.)

Rotations of a rigid body in 3D require 3 parameters to completely specify a specific rot $\mathcal{R}$.

Intuitively there are angles $\overbrace{(\theta, \varphi)}^{\hat{\Omega}}$ to give the new orientation of the original body-fixed $\hat{z}$ axis, and an angle χ that gives the rotation of the body itself about the new $\hat{\Omega}$ direction:



two equivalent choices in the description

- "body-fixed coord system" moves with the body.
- "space-fixed coord system" does not change.

Many choices for the three rigid-body angles are possible. The most often used in classical and quantum mechanics are the "Euler angles" (α, β, γ).

definition: 1st, recall: a general rotation about an axis $\hat{n}$ by θ in QM is given by

$$U(\theta\hat{\eta}) = e^{-i\theta\hat{\eta}\cdot\vec{J}}$$

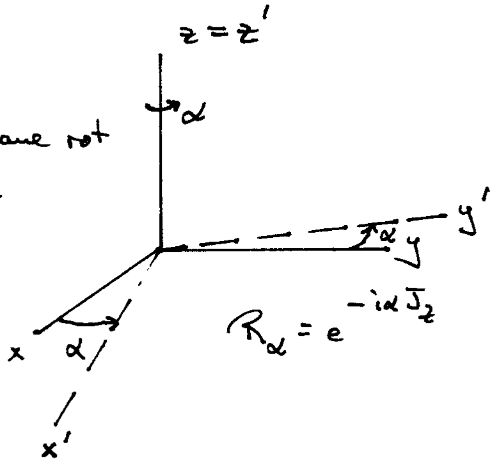
suppose we define the rot. by a three-stage transformation of the coordinate system:

1) rot. about $\hat{z}$ by α $\quad \mathcal{R}_\alpha = e^{-i\alpha J_z}$

2) rot. about the new $\hat{y}'$ axis by β $\quad \mathcal{R}_\beta = e^{-i\beta J_{y'}}$

3) finally, rot. about the $\hat{z}''$ axis by γ $\quad \mathcal{R}_\gamma = e^{-i\gamma J_{z''}}$

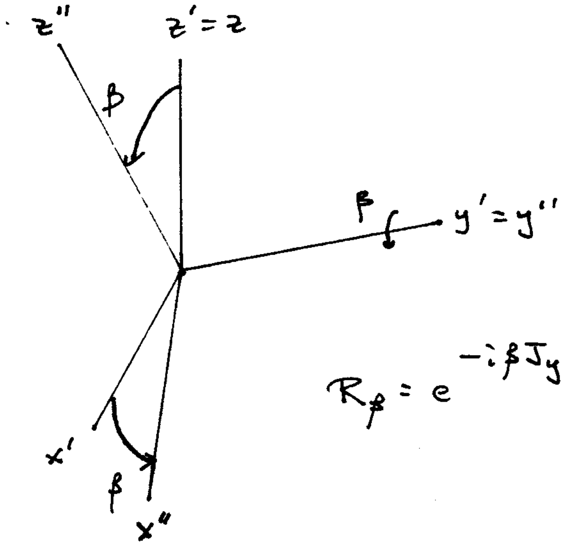
1. in-plane rot by α



$R_\alpha = e^{-i\alpha J_z}$

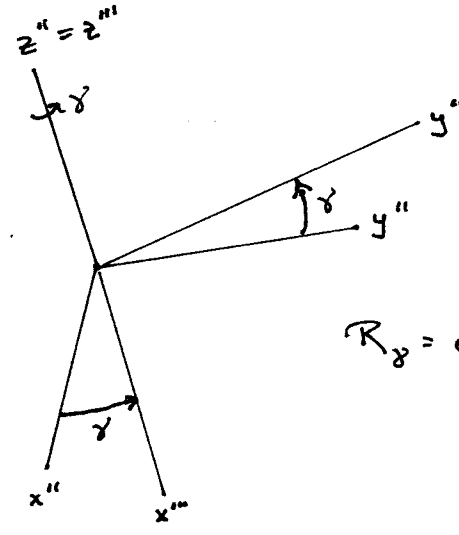
(still in plane)

2. "tip" about y' by β



$R_\beta = e^{-i\beta J_{y'}}$

3. rot about z'' by γ



$R_\gamma = e^{-i\gamma J_{z''}}$

Looks complicated, more so than (θ, φ, ψ), but it's easier to calculate with

$$R_{\alpha\beta\gamma} = \underbrace{e^{-i\gamma J_{z''}}}_{R_\gamma} \underbrace{e^{-i\beta J_{y'}}}_{R_\beta} \underbrace{e^{-i\alpha J_z}}_{R_\alpha}$$

1st, a ~~simplification~~ : we can simply relate these ′ and ″ coordinate rotations to rotations about the original x y z axes.

The rot. by β about y', $e^{-i\beta J_{y'}}$, is the same as

1. rot back to the original coord sys $(-\alpha)$
2. rot about the original y axis by β
3. return rot to the new coordinate sys (α).

So,

$$\boxed{\underbrace{e^{-i\beta J_{y'}}}_{\substack{R_\beta \\ \text{about } y'}} = \underbrace{e^{-i\alpha J_z}}_{\substack{3.\ R_{+\alpha} \\ \text{about } z}} \underbrace{e^{-i\beta J_y}}_{\substack{2.\ R_\beta \\ \text{about } y}} \underbrace{e^{+i\alpha J_z}}_{\substack{1.\ R_{-\alpha} \\ \text{about } z}}}$$

Similarly, the rot by γ about z'' can be "backed up" to a rotation by γ about the previous z',

$$\boxed{\underbrace{e^{-i\gamma J_{z''}}}_{\substack{R_\gamma \\ \text{about } z''}} = \underbrace{e^{-i\beta J_{y'}}}_{\substack{3.\ R_\beta \\ \text{about } y'}} \underbrace{e^{-i\gamma J_{z'}}}_{\substack{2.\ R_\gamma \\ \text{about } z'}} \underbrace{e^{+i\beta J_{y'}}}_{\substack{1.\ R_{-\beta} \\ \text{about } y'}}}$$

which gives

$$R_{\alpha\beta\gamma} = \underbrace{\left(e^{-i\beta J_{y'}} e^{-i\gamma J_{z'}} \cancel{e^{+i\beta J_{y'}}}\right)}_{R_\gamma} \underbrace{\cancel{e^{-i\beta J_{y'}}}}_{R_\beta} \underbrace{e^{-i\alpha J_z}}_{R_\alpha}$$

$$= \overbrace{\left(e^{-i\alpha J_z} e^{-i\beta J_y} \cancel{e^{+i\alpha J_z}}\right)} \underbrace{e^{-i\gamma J_{z'}}}_{\left(\cancel{e^{-i\alpha J_z}} e^{-i\gamma J_z} \cancel{e^{+i\alpha J_z}}\right)} \cancel{e^{-i\alpha J_z}}$$

$$\therefore \quad \underline{R_{\alpha\beta\gamma} = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}}$$

Thus the original rotation ≡ three rotations about space-fixed axes, if the order of rot. is inverted. Much easier for QM!

Rotation matrices

Now we can evaluate the effect of a general rotation $R_{\alpha\beta\gamma}$ on a quantum state $|jm\rangle$. Since total angular momentum $\vec{J}^2$ is conserved under rotations, we expect a linear combination of $\{|jm'\rangle\}$ basis states.

This defines the "rotation matrix" (Wigner)

$$\boxed{R_{\alpha\beta\gamma}\,|jm\rangle \equiv \sum_{m'} \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)\,|jm'\rangle}$$

The coefficients that make up the rotation matrix can "easily" be evaluated as matrix elements,

$$\mathcal{D}^{(j)}_{m'm}(\alpha\beta\gamma) = \langle jm'|R_{\alpha\beta\gamma}|jm\rangle = \langle jm'|\,e^{-i\alpha J_z}\,e^{-i\beta J_y}\,e^{-i\gamma J_z}\,|jm\rangle$$

$$= \underbrace{e^{-i\alpha m'}\,e^{-i\gamma m}}_{\text{simple phases for rot. about } z}\;\underbrace{\underbrace{\langle jm'|e^{-i\beta J_y}|jm\rangle}_{\text{nontrivial m.e.s.}}}_{\equiv\, d^{(j)}_{m'm}(\beta)}$$

these are real polynomials in $\cos(\beta/2)$ and $\sin(\beta/2)$

Wigner gives the d-functions as a series,

$$\boxed{d^{(j)}_{m'm}(\beta) = \left[(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!\right]^{1/2}\cdot\sum_k \frac{(-)^k}{(j-m'-k)!\,(j+m-k)!\,(k+m'-m)!\,k!}\cdot\cos(\beta/2)^{2j+m-m'-2k}\,\sin\left(\tfrac{\beta}{2}\right)^{m'-m+2k}\quad (m'\geq m)}$$

where the integer k runs over all values for which the factorials are nonsingular (≥ 0).

It may not help to know that this can be written as a ${}_2F_1$ hypergeometric function,

$$d^{(j)}_{m'm}(\beta) = \left[\frac{(j-m)!\,(j+m')!}{(j+m)!\,(j-m')!}\right]^{1/2} \frac{\cos\left(\frac{\beta}{2}\right)^{2j+m-m'}\left(-\sin\left(\frac{\beta}{2}\right)\right)^{m'-m}}{(m'-m)!}$$

$$\cdot\, {}_2F_1\left(m'-j,\,-m-j;\,m'-m+1 : -\tan^2\left(\tfrac{\beta}{2}\right)\right)$$

$(m' \geq m)$

there is also a closed-form result involving Jacobi polynomials

n.b.

$${}_2F_1(a,b:c:x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$(a)_n \equiv a(a+1)\ldots(a+n-1)$

and $(a)_0 \equiv 1$

<u>Some simple properties of the d-functions and examples</u>

$$\left.\begin{aligned} d^{(j)}_{m'm}(-\beta) &= (-)^{m'-m}\, d^{(j)}_{m'm}(\beta) \\ \underbrace{d^{(j)}_{m'm}(-\beta) = d^{(j)}_{mm'}(\beta)} \end{aligned}\right\} \Longrightarrow d^{(j)}_{m'm}(\beta) = (-)^{m'-m}\, d^{(j)}_{mm'}(\beta)$$

This is because a rot by $-\beta$ gives the transpose of the $+\beta$ rot. matrix.

(inverse rot, $R_{-\beta} = R_\beta^{-1}$) $\qquad (R^T = R^{-1})$ (matrices)

the closed form result implies

$$d^{(j)}_{m'm}(\beta) = d^{(j)}_{-m,-m'}(\beta) \qquad\qquad d^{(j)}_{m'm}(\beta) = (-)^{m'-m}\, d^{(j)}_{-m',-m}(\beta)$$

(can use this for $m' < m$ on ${}_2F_1$ formula above)

Simple properties from def^n of rot^s :

Note for

$$R_{\alpha\beta\gamma} = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z}$$

the inverse rotation is simply

$$R^{-1}_{\alpha\beta\gamma} = e^{+i\gamma J_z}\, e^{+i\beta J_y}\, e^{+i\alpha J_z}$$

and since $\mathcal{D}^{\dagger} = \mathcal{D}^{-1}$,

$$\underbrace{\langle jm' | R^{-1}_{\alpha\beta\gamma} | jm \rangle}_{} = \langle jm | R_{\alpha\beta\gamma} | jm' \rangle^{*}$$

$$\underbrace{\mathcal{D}^{(j)}_{m'm}(-\gamma, -\beta, -\alpha)}_{} = \mathcal{D}^{(j)}_{mm'}(\alpha, \beta, \gamma)^{*}$$

$$e^{+im'\gamma}\, e^{+im\alpha}\, \underbrace{d^{(j)}_{m'm}(-\beta)}_{} = e^{+i\alpha m}\, e^{+i\gamma m'}\, \underbrace{d^{(j)}_{mm'}(\beta)}_{}$$

$$d^{(j)}_{m'm}(-\beta) = d^{(j)}_{mm'}(\beta)$$

which was given previously

Explicit examples of d-functions

1. Stupid one $\underset{\sim}{j=0}$, $d^{(0)}_{00}(\beta) = e^{-i\beta\cdot I} = 1$

2. $\underline{j=1/2}$ $\quad d^{(1/2)}_{m'm}(\beta) = \langle \tfrac{1}{2}\, m' | \underbrace{e^{-i\beta J_y}}_{} | \tfrac{1}{2}\, m\rangle$, $J_y = \frac{1}{2}\sigma_y$

$$\begin{bmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{bmatrix}$$

$\therefore\ d^{(1/2)}_{1/2\,1/2}(\beta) = \cos(\beta/2)$ $\qquad d^{(1/2)}_{+1/2\,-1/2}(\beta) = -\sin(\beta/2)$

$d^{(1/2)}_{-1/2\,1/2}(\beta) = +\sin(\beta/2)$ $\qquad d^{(1/2)}_{(-1/2\,-1/2)}(\beta) = \cos(\beta/2)$

3. $\underset{\sim}{j=1}$

$$d^{(1)}_{m'm}(\beta) = \begin{bmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ +\frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & +\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{bmatrix}$$

(can get this e.g. by exponentiating the $\underset{\sim}{3}$ rep of J_y)

results (special cases) for general l (integer j):

We can build up $\mathcal{D}^{(j)}_{m'm}$ for arbitrarily large j using a CG series (next).

1st, two more special cases: $\quad d^{(l)}_{00}(\beta) = P_l(\cos\theta)$

$$\mathcal{D}^{(l)}_{m0}(\alpha,\beta,-) = \left[\frac{4\pi}{2l+1}\right]^{1/2} Y^{*}_{lm}(\beta,\alpha)$$

Products of rotation matrices $\mathcal{D}\cdot\mathcal{D}$

These arise e.g. when we consider rotations of composite states. Recall from the CG series for a product state

$$\underline{|jm\rangle = \sum_{m_1 m_2} \underbrace{\langle j_1 m_1, j_2 m_2 | jm\rangle}_{\text{CG coeffs}} |j_1 m_1\rangle |j_2 m_2\rangle}$$

Now suppose we apply a general $R_{\alpha\beta\gamma}$ rotation to this state:

$$\underline{R_{\alpha\beta\gamma}|jm\rangle} = \underline{\sum_{m'} \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)\,|jm'\rangle}$$

$$= \sum_{m_1,m_2} \langle j_1 m_1, j_2 m_2 | jm\rangle\, R_{\alpha\beta\gamma}\left\{|j_1 m_1\rangle\,|j_2 m_2\rangle\right\}$$

$$= \sum_{\substack{m_1 m_2\\ m_1' m_2'}} \langle j_1 m_1, j_2 m_2 | jm\rangle\, \mathcal{D}^{(j_1)}_{m_1' m_1}(\alpha,\beta,\gamma)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(\alpha,\beta,\gamma)\, |j_1 m_1'\rangle |j_2 m_2'\rangle$$

Pull out the 1st rotation matrix by multiplying by $\langle jm''|$, $\left(\begin{smallmatrix}\text{\& relabel}\\ m''\to m'\end{smallmatrix}\right)$;

$$\rightarrow \quad \mathcal{D}^{(j)}_{m'm}\underbrace{(\alpha,\beta,\gamma)}_{\equiv R} = \sum_{\substack{m_1 m_2\\ m_1' m_2'}} \langle j_1 m_1, j_2 m_2 | jm\rangle\, \mathcal{D}^{(j_1)}_{m_1' m_1}(R)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(R)\, \underbrace{\langle jm'|\cdot|j_1 m_1'\rangle|j_2 m_2'\rangle}_{\substack{\text{itself the CG coeff}\\ \langle jm'|j_1 m_1', j_2 m_2'\rangle}}$$

or

$$\mathcal{D}^{(j)}_{m'm}(R) = \underbrace{\sum_{\substack{m_1 m_2\\ m_1' m_2'}}}_{\substack{\text{note really only a}\\ \text{double sum: } m_2 = m - m_1,\\ m_2' = m' - m_1'}} \langle j_1 m_1, j_2 m_2 | jm\rangle \langle jm' | j_1 m_1', j_2 m_2'\rangle \cdot \mathcal{D}^{(j_1)}_{m_1' m_1}(R)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(R)$$

This can be used to construct all higher-j rot. matrices e.g. starting from $j_1, j_2 = 1/2$!

since the magnetic phases are trivial, this is really an identity for the nontrivial d-functions

$$\mathscr{D}^{(j)}_{m'm}(R) = e^{-i\alpha m'} e^{-i\gamma m} \underbrace{d^{(j)}_{m'm}(\beta)}$$

<u>an e.g.</u> <u>Let's try it and see.</u> what is $d^{(1)}_{00}(\beta)$?

$$d^{(1)}_{00}(\beta) = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle \tfrac12 m_1, \tfrac12 m_2 | 10 \rangle \langle 10 | \tfrac12 m_1', \tfrac12 m_2' \rangle \, d^{(1/2)}_{m_1' m_1}(\beta)\, d^{(1/2)}_{m_2' m_2}(\beta)$$

n.b.

$|10\rangle = \frac{1}{\sqrt2} |\tfrac12 \tfrac12\rangle |\tfrac12 -\tfrac12\rangle + \frac{1}{\sqrt2} |\tfrac12 -\tfrac12\rangle |\tfrac12 \tfrac12\rangle$

$\therefore$ all CG coeffs shown here are $+1/\sqrt2$.

$$= \underbrace{\langle \tfrac12 \tfrac12, \tfrac12 -\tfrac12 | 10 \rangle^2}_{1/2} \cdot \underbrace{d^{(1/2)}_{1/2\,1/2}(\beta)}_{c} \underbrace{d^{(1/2)}_{-1/2\,-1/2}(\beta)}_{c}$$

$$+ \underbrace{\langle \tfrac12 -\tfrac12, \tfrac12 \tfrac12 | 10 \rangle^2}_{1/2} \cdot \underbrace{d^{(1/2)}_{-1/2\,-1/2}(\beta)}_{c} \cdot \underbrace{d^{(1/2)}_{1/2\,1/2}(\beta)}_{c}$$

$$+ \underbrace{\langle \tfrac12 \tfrac12, \tfrac12 -\tfrac12 | 10 \rangle \cdot \langle 10 | \tfrac12 -\tfrac12, \tfrac12 \tfrac12 \rangle}_{1/2} \cdot \underbrace{d^{(1/2)}_{-1/2\,1/2}(\beta)}_{s} \cdot \underbrace{d^{(1/2)}_{1/2\,-1/2}(\beta)}_{-s}$$

$$+ \underbrace{\langle \tfrac12 -\tfrac12, \tfrac12 \tfrac12 | 10 \rangle \cdot \langle 10 | \tfrac12 \tfrac12, \tfrac12 -\tfrac12 \rangle}_{1/2} \cdot \underbrace{d^{(1/2)}_{1/2\,-1/2}(\beta)}_{-s} \cdot \underbrace{d^{(1/2)}_{-1/2\,1/2}(\beta)}_{s}$$

$c \equiv \cos(\beta/2)$
$s \equiv \sin(\beta/2)$

$$= \frac12 \left\{ 2c^2 - 2s^2 \right\} = \cos^2(\beta/2) - \sin^2(\beta/2) = \underbrace{\cos(\beta)} \quad ✓$$

Can similarly build up <u>all</u> higher $d^{(j)}_{m'm}(\beta)$ functions, given the CG coefficients and $\{d^{(1/2)}_{m'm}(\beta)\}$.

Opposite direction : rot. of $|j_1\rangle|j_2\rangle$ states

$$|j_1 m_1\rangle|j_2 m_2\rangle = \sum_{\substack{j\, m\, (m=m_1+m_2)}} \langle jm|j_1 m_1, j_2 m_2\rangle |jm\rangle$$

apply a general rotation $R_{\alpha\beta\gamma}$

$$R\{|j_1 m_1\rangle|j_2 m_2\rangle\} = \sum_{m_1' m_2'} \mathcal{D}^{(j_1)}_{m_1' m_1}(R)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(R)\, |j_1 m_1'\rangle|j_2 m_2'\rangle$$

$$= \sum_{j\, \underbrace{m}_{=m_1+m_2,\text{ not really a sum}}\, m'} \langle jm|j_1 m_1, j_2 m_2\rangle\, \mathcal{D}^{(j)}_{m'm}(R)\, |jm'\rangle$$

dot $\langle j_1 m_1''|\langle j_2 m_2''|$ into this equation, (then reset $'' \to '$);

$$\mathcal{D}^{(j_1)}_{m_1' m_1}(R)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(R)$$

$$= \sum_{j\, m\, m'} \langle jm|j_1 m_1, j_2 m_2\rangle\, \mathcal{D}^{(j)}_{m'm}(R) \underbrace{\{\langle j_1 m_1'|\langle j_2 m_2'|\cdot|jm'\rangle\}}_{\text{CG coeff}}$$

$m' = m_1' + m_2'$, $m = m_1 + m_2$ } not really summed

$$= \sum_{\boxed{j}\, m\, m'} \langle jm|j_1 m_1, j_2 m_2\rangle \langle j_1 m_1', j_2 m_2'|jm'\rangle\, \mathcal{D}^{(j)}_{m'm}(R)$$

j: only real sum; $m\,m'$ $\to$ fixed

Thus the product of the entries of two rotation matrices, in general of different spin, can be written as a sum of individual $\mathcal{D}^{(j)}_{m'm}$ matrices, with a sum over j, from $|j_1 - j_2|$ to $j_1 + j_2$.

This is the CG series for rotation matrices.

What use is this $\mathcal{D}\cdot\mathcal{D}$ decomposition to us?

One surprising result: Gaunt formula!

Recall for integer j we stated

$$\mathcal{D}^{(l)}_{m0}(\alpha,\beta,\gamma) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y^{*}_{lm}(\beta,\alpha)$$

This implies, from prev $\mathcal{D}\cdot\mathcal{D}$ result,

$$\mathcal{D}^{(l_1)}_{m_1'0}(\alpha\beta\gamma)\,\mathcal{D}^{(l_2)}_{m_2'0}(\alpha\beta\gamma) = \frac{4\pi}{\sqrt{(2l_1+1)(2l_2+1)}}\, Y^{*}_{l_1m_1'}(\beta,\alpha)\,Y^{*}_{l_2m_2'}(\beta,\alpha)$$

$$= \sum_{l} \langle l m | l_1 0, l_2 0\rangle \cdot \langle l_1 m_1', l_2 m_2' | l m'\rangle \underbrace{\mathcal{D}^{(l)}_{m'0}(\alpha\beta\gamma)}_{\left(\frac{4\pi}{(2l+1)}\right)^{1/2} Y^{*}_{lm'}(\beta\alpha)}$$

$$\boxed{\therefore\ Y_{l_1m_1}(\Omega)\cdot Y_{l_2m_2}(\Omega) = \frac{1}{\sqrt{4\pi}} \sum_{\substack{l \\ (|l_1-l_2| \text{ to } l_1+l_2)}} \left[\frac{(2l_1+1)(2l_2+1)}{(2l+1)}\right]^{1/2} \cdot \langle l 0 | l_1 0, l_2 0\rangle \cdot \langle l_1 m_1, l_2 m_2 | l m\rangle \cdot \underbrace{Y_{lm}(\Omega)}_{m = m_1+m_2}}$$

This is the Gaunt formula, which was given w/o justification previously.

<u>Dynamical application – rotational levels of a rigid body in QM</u> (~ molecules)

The classical Hamiltonian for rot. of a rigid body is

$$H = \sum_{i=1}^{3} \frac{L_i^2}{2I_i}$$

$$\underbrace{i = \text{princ. axes} = a, b, c}_{\text{(always an orthogonal set)}}$$

For the QM version, factor an $\hbar$ out of each ang. mom. op. & we have

$$H_{op} = \sum_{i=1}^{3} \frac{\hbar^2}{2I_i} \hat{L}_i^2$$

special cases are the spherically symmetric top $I_a = I_b = I_c = I$

axially symm top $I_a = I_b = I$, $I_c = I' \neq I$

and asymm top, all different.

The 1st two cases are easy to solve in QM.

Note the angular momentum operators $\{\hat{L}_i\}$ refer to <u>body-fixed coordinates</u> (= rots. about a, b, c body-fixed axes).

The work is in rewriting these ops. in space-fixed coordinates, such as the Euler angles (α, β, γ), or our original set (θ, φ, χ).

In the latter coords this gives, for the axisymmetric top (Pauling + Wilson)

$$H_{op} = -\frac{\hbar^2}{2I}\left\{ \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2 + \left(\frac{1}{\tan^2\theta} + \frac{I}{I'}\right)\partial_\chi^2 - \frac{2\cos\theta}{\sin^2\theta}\partial^2_{\chi\varphi} \right\}$$

& we are to find a wavefunction $\boxed{\Psi(\theta, \varphi, \chi)}$ that satisfies — orientation amplitude

$$H_{op}\Psi = E\Psi.$$

This will give the spectrum of states for the symm. rigid rotator.

by inspection, the magnetic angles are plane waves,

$$\Psi(\Theta,\varphi,x) = e^{im\varphi}\, e^{ikx} \cdot f_{m,k}(\theta)$$

integers, each $0, \pm 1, \pm 2, \ldots$

$' \equiv \frac{d}{d\Theta}$

$$-\frac{\hbar^2}{2I}\left\{ f'' + \cot(\theta) f' - \frac{m^2}{\sin^2(\theta)} f - \left(\frac{I}{I'} + \frac{1}{\tan^2(\theta)}\right) f + \frac{2mk\cos(\theta)}{\sin^2(\theta)} f \right\} = Ef$$

the solution of this DE is $f = d^{(\ell)}_{m,k}(\theta)$,

$$\boxed{\Psi(\theta,\varphi,x) = \eta\, e^{im\varphi}\, e^{ikx}\, d^{(\ell)}_{m,k}(\theta)}$$

(not normalized yet)

and the resulting energies are

$$E = \frac{\hbar^2}{2I}\ell(\ell+1) + \frac{\hbar^2}{2}\left(I'^{-1} - I^{-1}\right) k^2$$

This can be seen more intuitively using

$$H = \frac{\hbar^2}{2I}\left(\hat{L}_a^2 + \hat{L}_b^2\right) + \frac{\hbar^2}{2I'}\hat{L}_c^2 = \frac{\hbar^2}{2I}\underbrace{\hat{\vec{L}}^2}_{\text{also } \hat{\vec{L}}^2 \text{ space-fixed, } \ell(\ell+1) \text{ eigenvalue}} + \frac{\hbar^2}{2}\left(I'^{-1} - I^{-1}\right)\underbrace{\hat{L}_c^2}_{k^2 \text{ on } e^{ikx}}$$

$$\therefore\; E = \frac{\hbar^2}{2I}\ell(\ell+1) + \frac{\hbar^2}{2}\left(\frac{1}{I'} - \frac{1}{I}\right)k^2 \text{ is expected anyway!}$$