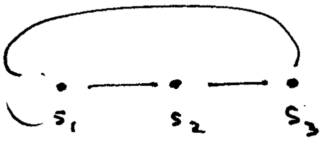
How are groups used in physics, esp QM?

Schur's lemma, disguised as physics (Actually Lemma II.)

If the Hamiltonian H of a system commutes with all the elements of a symmetry group, then the basis vectors which span a given irreducible rep all have the same energy eigenvalue.

$$\boxed{[H, g] = 0_{\forall g \in \mathcal{G}} \longrightarrow H = \lambda I \text{ operating on states } \{|\psi\rangle\} \text{ that span an irrep of } \mathcal{G}.}$$

Specific example we have already done as a homework problem:
3 spins interacting through a Heisenberg afm coupling



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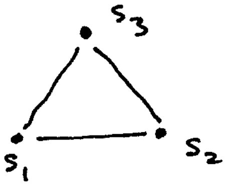


$$H = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1$$

What is the symmetry group of this system? Clearly we can exchange any pair of spins & will get the same H. [We can also permute cyclically with (123) and (132) & leave H invariant]. Thus the symm group is at least S_3 [actually its larger – rotations], so states which span an irrep of S_3 will be degenerate.

Irreps of S_3 are $\underset{\underline{1}}{\mathcal{D}_1}$ $\underset{\underline{1}}{\mathcal{D}_2}$ and $\underset{\underline{2}}{\mathcal{D}_3}$, so barring "accidents" (e.g. additional symm. (rots!)) we expect levels to be either nondegenerate singlets

or twofold degenerate doublets.

Lets start with a complete basis for some sector and see which irreps it contains.

$S_z = +\frac{1}{2}$ $\qquad |{+}{+}{-}\rangle \qquad |{+}{-}{+}\rangle \qquad |{-}{+}{+}\rangle$

3 basis states – will give $\underline{3}$ rep, must be reducible.

$\underline{S_3}$ Look at effect of one element from each class

$\mathbb{C}_1 = \{e\}$
$\mathbb{C}_2 = \{(123), (132)\}$
$\mathbb{C}_3 = \{(12), (13), (23)\}$

what are we doing?
getting $\mathcal{D}(g)$ from

$$\underbrace{\mathcal{O}_g |\psi_\mu\rangle} = \sum_\nu \underbrace{\mathcal{D}_{\nu\mu}(g)}_{\overset{\shortparallel}{\mathcal{D}^T_{\mu\nu}(g)}} \underbrace{|\psi_\nu\rangle}$$

$$e \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix}$$

$$\mathcal{D}(e) \longrightarrow \chi(e) = \mathrm{Tr}\left[\mathcal{D}(e)\right] = 3$$

(same as $\mathcal{D}^T$) $\qquad \chi(e) \equiv \chi(C_1)$

$$(12) \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix} = \begin{bmatrix} |{+}{+}{-}\rangle \\ |{-}{+}{+}\rangle \\ |{+}{-}{+}\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix}$$

$$\mathcal{D}((12)) \longrightarrow \chi((12)) = 1$$

(same as $\mathcal{D}^T$) $\qquad \chi((12)) \equiv \chi(C_3) \hookrightarrow$ to agree w/ M+W

$$(123) \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix} = \begin{bmatrix} |{-}{+}{+}\rangle \\ |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \end{bmatrix}$$

$$\mathcal{D}^T((123)) \longrightarrow \chi((123)) = 0$$

$\chi((123)) = \chi(C_2)$

element e.g.	Class	$\chi(c)$
e	$\mathbb{C}_1$	3
(123)	$\mathbb{C}_2$	0
(12)	$\mathbb{C}_3$	1

now recall our ~~class~~ table for S_3:

Class	$\chi_1(c)$	$\chi_2(c)$	$\chi_3(c)$
$\mathbb{C}_1$	1	1	2
$\mathbb{C}_2$	1	1	-1
$\mathbb{C}_3$	1	-1	0

To see how many times the ith irrep of S_3 is present in $\mathcal{D}$, note if

$$\mathcal{D}(g) = c_1 \mathcal{D}_1 \oplus c_2 \mathcal{D}_2 \oplus c_3 \mathcal{D}_3$$

then

$$\chi(\mathbb{C}_k) = c_1 \chi_1(\mathbb{C}_k) + c_2 \chi_2(\mathbb{C}_k) + c_3 \chi_3(\mathbb{C}_k)$$

so we can use the orthog relation for characters of irreps,

$$\sum_{\substack{k=1 \\ \text{sum over} \\ \text{classes}}}^{\mathbb{K}} \overset{\text{number of elements in } k^{\text{th}} \text{ class}}{p_k} \underbrace{\chi_{r_A}}_{\substack{\text{irrep} \\ A}}(\mathbb{C}_k)^* \underbrace{\chi_{r_B}}_{\substack{\text{irrep} \\ B}}(\mathbb{C}_k) = n\,\delta_{AB}$$

to solve for c_i,

$$\boxed{c_{r_A} = \frac{1}{n} \sum_{k=1}^{\mathbb{K}} p_k \chi_{r_A}(\mathbb{C}_k)^* \overset{\text{our rep.}}{\chi(\mathbb{C}_k)}}$$

very important: how to determine irrep. content

Our case 3 classes,
$p_1 = 1$ (class of e)
$p_2 = 2$ (class of (123))
$p_3 = 3$ (class of (12))

irreps of S_3
r_1 = ident rep
r_2 = antisymmetric rep
r_3 = 2×2 matrix rep

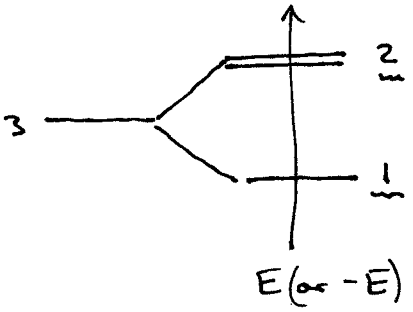
So

$$c_{r_1} = \frac{1}{6}\left[\overset{p_1}{1} \cdot \overset{\chi_{r_1}(\mathbb{C}_1)^*}{1} \cdot \overset{\chi(\mathbb{C}_1)\text{ our }3\times3\text{ rep}}{3} + 2 \cdot 1 \cdot (0) + 3 \cdot 1 \cdot 1 \right] = 1 \quad \text{one ident rep present}$$

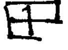
$$c_{r_2} = \frac{1}{6}\left[1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3(-1) \cdot (+1) \right] = 0 \quad \text{no antis. rep present}$$

$$c_{r_3} = \frac{1}{6}\left[1 \cdot 2 \cdot 3 + 2 \overset{(-1)}{\cdot} 0 + 3 \cdot 0 \cdot 1 \right] = 1, \quad \text{one 2 rep present.}$$

Since $[H, g] = 0$ implies that basis vectors spanning ireps are degenerate, we expect our $\underset{\sim}{3}$ space to split up into



} typical group theory argument: we know what degeneracies are present but know nothing about the real eigenenergies.

There is a way of constructing the basis vectors explicitly using "Young tableau" □, e.g.  for 3 spins, this tells you how to build the eigenvectors from "symmetrizers" acting on individual basis states like $|++-\rangle$. (See Hamermesh.) (We will discuss this subsequently.)
The basis vectors that span our $S_z = 1/2$ ireps are

$$\frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle\right) \equiv |1\rangle_{\underset{\sim}{1}}$$

(check e, (12), (123),... each gives this vector back when you operate on it.

For the 2D rep they may be taken to be

$$\frac{1}{\sqrt{3}}\left(|++-\rangle + \eta|+-+\rangle + \eta^2|-++\rangle\right) \equiv |1\rangle_{\underset{\sim}{2}} \qquad \eta = e^{2\pi i/3}$$

$$\text{so } \eta^3 = 1$$

$$\frac{1}{\sqrt{3}}\left(|++-\rangle + \underbrace{\eta^*}_{\eta^2}|+-+\rangle + \underbrace{\eta^{*2}}_{\eta^4=\eta}|-++\rangle\right) \equiv |2\rangle_{\underset{\sim}{2}}$$

They are orthonormal as you may confirm. Also

$$e\begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{D}^{(2)}(e) \text{ in } \underset{\sim}{2} \text{ rep.}}\begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix} \quad , \chi(e) = 2 \checkmark$$

$$(12)\,|1\rangle = \frac{1}{\sqrt{3}}\Big(|{+}{+}{-}\rangle + \eta\,|{-}{+}{+}\rangle + \eta^2\,|{+}{-}{+}\rangle\Big) = |2\rangle$$

and $(12)\,|2\rangle = |1\rangle$, so $\mathcal{D}^{(2)}((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in $\underset{\sim}{2}$ irrep,

$\chi((12)) = 0$ ✓

$$(123)\,|1\rangle = \frac{1}{\sqrt{3}}\Big(|{-}{+}{+}\rangle + \eta\,|{+}{+}{-}\rangle + \eta^2\,|{+}{-}{+}\rangle\Big) = \eta\,|1\rangle$$

$$(123)\,|2\rangle = \frac{1}{\sqrt{3}}\Big(|{-}{+}{+}\rangle + \eta^*\,|{+}{+}{-}\rangle + \eta^{*2}\,|{+}{-}{+}\rangle\Big) = \eta^*\,|2\rangle,$$

so $\mathcal{D}^{(2)}((123)) = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{bmatrix}$ in this basis, and so

$$\chi((123)) = \mathrm{Tr}[\,{}''\,] = 2\cos\frac{2\pi}{3} = -1 \;✓$$

Since these basis vectors span irreps, of a symm group of H, H is guaranteed to be $\propto E_r\, I$ on <u>all</u> the basis vectors of rep "r".

If you <u>really</u> want to know E for this little example:
The actual H eigenvalues can be found by a simple trick, $\underbrace{\vec{S}^{\,2}_{tot}}_{S_T(S_T+1)} = (\overrightarrow{S_1+S_2+S_3})^2 = 3\cdot\frac{3}{4} + 2H$

$$\therefore E = \frac{1}{2}S_T(S_T+1) - \frac{9}{8} = \begin{cases} \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2} - \frac{9}{8} = \underline{\frac{3}{4}} \\ \cdot\frac{3}{8} - \frac{9}{8} = \underline{-\frac{3}{4}} \end{cases}$$

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \underline{\frac{3}{2}}\oplus\underline{\frac{1}{2}}\oplus\frac{1}{2}$$

Which is which? Symm states have highest spin, $\frac{1}{\sqrt{3}}\Big(|{+}{+}{-}\rangle + |{+}{-}{+}\rangle + |{-}{+}{+}\rangle\Big)$ is $|S_T = 3/2, S_T^z = 1/2\rangle$

$E = 3/4$

others have $E = -3/4$
and are part of 2 sep spin-½ multiplets

All states of 3 spin-1/2 particles:

$S_T = 3/2$

$S_T^z = 3/2$: $\boxed{|+++\rangle}$ $\perp$

$1/2$: $\boxed{\frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle\right)}$ $\perp$

$-1/2$: $\boxed{\frac{1}{\sqrt{3}}\left(|+--\rangle + |-+-\rangle + |--+\rangle\right)}$ $\perp$

$-3/2$: $\boxed{|---\rangle}$ $\perp$

$$\overbrace{\begin{matrix}\frac{1}{\sqrt{3}}\left(|++-\rangle + \eta|+-+\rangle + \eta^2|-++\rangle\right) \\ \frac{1}{\sqrt{3}}\left(|+--\rangle + \eta|--+\rangle + \eta^2|-+-\rangle\right)\end{matrix}}^{S_T = \frac{1}{2}^{(1)}} \;;\; \overbrace{\begin{matrix}\frac{1}{\sqrt{3}}\left(|++-\rangle + \eta^*|+-+\rangle + \eta^{*2}|-++\rangle\right) \\ \frac{1}{\sqrt{3}}\left(|+--\rangle + \eta^*|--+\rangle + \eta^{*2}|-+-\rangle\right)\end{matrix}}^{S_T = \frac{1}{2}^{(2)}}$$

$\underset{\sim}{2}$ $\underset{\sim}{2}$

$\boxed{}$ = spin $\overset{\text{irred.}}{\text{multiplets}}$ (SU(2)) (obvious but not discussed here)

$\boxed{}$ [illegible]

$\therefore$ full symmetry group of this problem is

$$\mathcal{G} = SU(2) \otimes S_3$$

$\vec{S}_i \cdot \vec{S}_j$: rotation symm of spin vectors $\vec{S}_i$ (no site hopping) $\uparrow$ SU(2)

$\therefore$ total spin is conserved (a good quantum number)

permutation symmetry of the 3 sites. $\uparrow$ S_3

if e.g. $J_{12} \neq J_{13} = J_{23}$, we would lose S_3 symmetry and the two $S = 1/2$ multiplets would no longer be degenerate