

PHY522; Quantum Mechanics II, Problem Set 6

Due Mon. 26 Mar 2007 at the beginning of class.

1. Choice of picture in 1D quantum mechanics.

Many time dependence problems in conventional quantum mechanics are surprisingly easy to solve using the less well known Heisenberg picture, as this problem will hopefully illustrate.

Consider a 1D simple harmonic oscillator (SHO) with the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{k}{2}x^2. \quad (1)$$

In the Heisenberg picture the position and momentum operators in this SHO problem evolve with time, from $x_{op}(t=0) = x$ and $p_{op}(t=0) = p$ into more general forms $x_{op}(t)$ and $p_{op}(t)$.

a) (3 pts) Use the nested commutator approach to find $x_{op}(t)$ and $p_{op}(t)$ for this problem. Your results should be of the form $x_{op}(t) = f_1(t)x + f_2(t)p$ and $p_{op}(t) = g_1(t)p + g_2(t)x$.

b) (2 pts) Show using these results that the commutator $[x_{op}(t), p_{op}(t)]$ is time independent, and $= +i\hbar$ as you might expect.

c) (2 pts) Show also that $H_{op} = \frac{1}{2m}p_{op}(t)^2 + \frac{k}{2}x_{op}(t)^2$ is actually a static operator, as it must be ($[H, H] = 0$).

d) (3 pts) If a particle in this 1D SHO problem initially (at $t = 0$) has expected position $\langle \psi | x_{op}(0) | \psi \rangle = x_0 \neq 0$ and momentum $\langle \psi | p_{op}(0) | \psi \rangle = p_0 = 0$, what is the expected position $\langle \psi | x_{op}(t) | \psi \rangle$ and expected momentum $\langle \psi | p_{op}(t) | \psi \rangle$ at a general later time t . (Note that you have found these quite general results for the time evolution of an implicit initial wavefunction without ever writing an explicit wavefunction!)

2. Spin algebra in the Heisenberg picture.

In calculating the time dependence of the Heisenberg picture spin operator $\vec{S}_{op}(t)$ of a spin-1/2 particle in a static magnetic field, we had to evaluate the rather complicated looking expression

$$\vec{S}_{op}(t) = U^\dagger(t, 0) \frac{\vec{\sigma}}{2} U(t, 0) = \exp\left\{-i\frac{\mu_B t}{\hbar} \vec{\sigma} \cdot \vec{B}\right\} \frac{\vec{\sigma}}{2} \exp\left\{+i\frac{\mu_B t}{\hbar} \vec{\sigma} \cdot \vec{B}\right\}. \quad (2)$$

a) (3 pts) It is clearly useful to abbreviate this as

$$\vec{S}_{op}(t) = \exp\{-i\vec{\sigma} \cdot \vec{b}\} \frac{\vec{\sigma}}{2} \exp\{+i\vec{\sigma} \cdot \vec{b}\}. \quad (3)$$

Using formal algebraic properties of the Pauli matrices (not explicit matrices), use Eq.(3) to evaluate $\vec{S}_{op}(t)$. On resubstituting for \vec{B} , you will find that one part of $\vec{S}_{op}(t)$ is static and another rotates with the Larmor frequency (with mean vector zero). (Show these components on a diagram.)

b) (3 pts) Since I and $\vec{\sigma}$ form a complete basis for any 2×2 matrix, including $U(t, 0) = \exp\{+i\vec{\sigma} \cdot \vec{b}\}$, it is possible to write this matrix in the form $c_0 I + \vec{\sigma} \cdot \vec{c}$. Find c_0 and \vec{c} in terms of \vec{b} .

c) (4 pts) Using the new form $U(t, 0) = c_0 I + \vec{\sigma} \cdot \vec{c}$ from part b), evaluate $\vec{S}_{op}(t)$ and confirm that you recover the result of part a).

3. Exciting H atoms: time dependent perturbation theory.

In class I gave a rather bad example of time dependent perturbation theory, in which I assumed that a square pulse electric field was applied to a hydrogen atom in its ground state. The perturbing Hamiltonian was taken to be

$$H_I(t) = -eEz f(t) \quad (4)$$

where $E = 10$ [kV/cm] and $f(t) = \theta(T - t)\theta(t)$ (a square pulse of duration T) with $T = 1$ [μ sec]. The problem was to find the excitation probability of $1S \rightarrow 2P$, but this is unrealistic because the answer was driven by the discontinuity in the pulse function $f(t)$.

(10 pts) In a less pathological version of this problem we might consider driving the H atom using an extremely fast Gaussian pulse $f(t) = \exp(-t^2/T^2)$, where $T = 1$ [psec]. What is the numerical probability of the transition $1S \rightarrow 2P$ in this case? (In this problem only $f(t)$ is modified, the peak field strength remains $E = 10$ [kV/cm].)