PHY522; Quantum Mechanics II, Problem Set 4

Due Wednesday 14 Feb 2007 at the beginning of class.

Generic elastic scattering conventions: The incident particle has wavevector $\vec{k}$, the scattered wavevector is $\vec{k}'$, the scattering is elastic so $|\vec{k}| = |\vec{k}'| = k$, the momentum transfer wavevector is $\vec{q} = \vec{k}' - \vec{k}$, and the scattering angle between $\vec{k}$ and $\vec{k}'$ is θ.

1. Born approximation phase shifts, 3D spherical potential.

We found in class that the Born-order phase shifts $\{\delta_\ell\}$ for scattering from a spherical potential $V(r)$ in 3D are given by

$$\delta_\ell = -\frac{2mk}{\hbar^2} \int_0^\infty V(r)\, j_\ell(kr)^2\, r^2\, dr\ . \tag{1}$$

Consider the specific case of a spherical potential of height V_0 and radius R,

$$V(r) = V_0\, \theta(R - r)\ . \tag{2}$$

a) (*3 pts*) Evaluate this phase shift for S-wave scattering, and show that the result is

$$\delta_0 = -\frac{mV_0R^2}{\hbar^2}\ \frac{[1 - j_0(2\chi)]}{\chi}\ , \tag{3}$$

where $\chi = kR$ is the scaled momentum.

b) (*2 pts*) Plot this Born-order S-wave phase shift versus (scaled) momentum χ over the range $\chi \in [0, 10]$, for $mV_0R^2/\hbar^2 = 0.1$ and 1.0. (Note: a plausible condition for accuracy of this first Born approximation is $|\delta_0| << 1$.)

c) (*3 pts*) Using the small-argument limit of $j_\ell(kr)$ in Eq.(1), evaluate all the Born-order phase shifts $\{\delta_\ell\}$ in the near-threshold regime $kR << 1$.

d) (*2 pts*) These near-threshold phase shifts decrease rapidly in magnitude with increasing ℓ. Using Stirling's formula, find the asymptotic behavior of $\{\delta_\ell\}$ in the double limit $kR << 1$ and $\ell >> 0$.

2. Exact 3D spherical potential S-wave phase shift.

We may check the accuracy of the Born approximation in problem 1 by solving for the S-wave phase shift exactly. The interior and exterior radial wavefunctions are given by

$$R_{0,k}(r) = \begin{cases} j_0(pr), & r < R \\ A_0 j_0(kr) + B_0 \eta_0(kr), & r > R \end{cases} \tag{4}$$

where as usual outside the potential $E = \hbar^2 k^2/2m$ and in the interior, $E - V_0 = \hbar^2 p^2/2m$. (We are assuming $E > V_0$, so the interior solution is oscillatory.)

a) (*8 pts*) To find the exact S-wave phase shift, first impose the boundary conditions of $R(r)$ and $R'(r)$ continuity at $r = R$. (Sorry for the use of R for both radius and radial wavefunction.) This gives you the coefficients A_0 and B_0. Their ratio gives the phase shift, using

$$\delta_0 = -\tan^{-1}\left(B_0/A_0\right) . \tag{5}$$

The formula is complicated. You may simplify it somewhat using the fact that $j_1(x)\eta_0(x) - j_0(x)\eta_1(x) = 1/x^2$. The identities $dj_0(x)/dx = -j_1(x)$ and $d\eta_0(x)/dx = -\eta_1(x)$ are also useful.

b) (*2 pts*) Plot this exact phase shifts for the same parameters as in problem 1b. Note where the Born approximation is accurate and where it shows signs of failing.

You may be curious about the approach to the hard sphere limit, $V_0 \to \infty$. To study this limit we would have to rederive the phase shift with a modified spherical Bessel function for the internal wavefunction, $R_{0,k}(r) = i_0(\tilde{p}r)$ for $r < R$, because $V_0 > E$. This might make a good quantum project.

3. S-matrix.

As a generic S-matrix is an $n \times n$ unitary matrix, it is useful to consider some of their properties. Either prove or disprove the following assertions about $n \times n$ unitary matrices:

a) (*2 pts*) They all have determinant 1.

b) (*2 pts*) The eigenvalues are all pure phases.

c) (*2 pts*) The sum of all n phases is 0.

d) (*2 pts*) A general unitary matrix S has $n(n+1)/2$ free parameters.

e) (*2 pts*) If we write $S = e^{iG}$, the matrix G is Hermetian and traceless.