PHY522; Quantum Mechanics II, Problem Set 2

Due Wednesday 31 Jan 2007 at the beginning of class.

1. Green's function for the 3D free Schrödinger equation.

The Green's function (or "kernel") $K(\vec{x}-\vec{x}')$ for the free Schrödinger equation in n space dimensions satisfies

$$\left(E+\frac{\hbar^2}{2m}\nabla^2\right)^{-1} f(\vec{x}\,) \equiv \int d^n x' K(\vec{x}-\vec{x}\,')\, f(\vec{x}\,')\ . \tag{1}$$

a) (*3 pts*) Show that this equation is satisfied provided that

$$\left(E+\frac{\hbar^2}{2m}\nabla^2\right) K(\vec{x}-\vec{x}') = \delta(\vec{x}-\vec{x}')\ . \tag{2}$$

b) (*3 pts*) Introduce a Fourier transform for the kernel K,

$$K(\vec{x}-\vec{x}\,') = \frac{1}{(2\pi)^n}\int d^n k\ K(\vec{k}\,)\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\ , \tag{3}$$

and show that it satisfies

$$K(\vec{k}\ ) = \frac{1}{E-\hbar^2\vec{k}^2/2m}\ . \tag{4}$$

c) (*4 pts*) Using this $K(\vec{k}\ )$, evaluate Eq.(3) explicitly in 3D and show that the kernel is given by

$$K(\vec{x}-\vec{x}\,') = -\frac{m}{2\pi\hbar^2}\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \tag{5}$$

as claimed in class. (For simplicity you may wish to set $\vec{x} = r\hat{z}$ and $\vec{x}\,' = 0$ before doing the $d\Omega_k$ angular integral.) Note that after the angular integral you encounter two poles on the real axis in the contour integral over complex $k = |\vec{k}|$, and the choice of which pole to enclose is determined by our requirement of an outgoing spherical wave, e^{+ikr}/r not e^{-ikr}/r.

2. 1D delta-shell potential phase shifts.

In the previous problem set we considered barrier penetration through the "delta-shell potential", consisting of two delta functions at $x = 0, R$ of strength v_0,

$$V(x) = v_0 \big(\delta(x) + \delta(x - R)\big) \ . \tag{6}$$

The transmission amplitude we found for this problem was

$$t = \left\{1 + i\frac{u}{\chi} + \frac{u^2}{4\chi^2}\left(e^{2i\chi} - 1\right)\right\}^{-1} \tag{7}$$

where the scaled potential strength and particle momentum are $u = 2mv_0R/\hbar^2$ and $\chi = kR$. This amplitude can be written as a magnitude times a phase shift, $t = |t|\, e^{i\delta_t}$.

a) (*5 pts*) Solve for the transmission phase shift δ_t.

b) (*5 pts*) Plot δ_t over the range $\chi \in [0, 20]$ for $u = 1., 10., 100.$ (Recall that this should evolve continuously towards $\delta_t = 0$ in the high energy limit.) Note the correspondence between the motion of this phase shift and the positions and widths of the "resonances" we observed in the transmission probability $|t|^2$.

3. QM virial theorem.

Consider the 3D Schrödinger problem of a particle of mass m bound in a radial power-law potential,

$$V(r) = g\, r^\nu \ . \tag{8}$$

a) (*3 pts*) Using dimensional arguments, show that all bound state energies must scale as

$$E_n = n_n \cdot \left(\frac{\hbar^2}{m}\right)^{\nu/(\nu+2)} g^{2/(\nu+2)} \tag{9}$$

where n_n is a pure number.

b) (*3 pts*) Use the Feynman-Hellmann theorem combined with Eq.(9) to find the expected values of the kinetic and potential energies, $\langle n|T|n\rangle = \langle n|(\vec{p}^{\,2}/2m)|n\rangle$ and $\langle n|V|n\rangle = \langle n|gr^\nu|n\rangle$.

c) (*4 pts*) Use your results in part b) to find the virial theorem for this potential, which states that the expected value of the ratio of potential and kinetic energies

$$\frac{\langle n|V|n\rangle}{\langle n|T|n\rangle} \tag{10}$$

is independent of the state n, the mass m and the potential strength g.