PHY522; Quantum Mechanics II, Problem Set 1

Due Wed. 24 Jan 2007 at the beginning of class.

1. The simplest 1D scattering problem.

A beam of free particles with momentum k (described by the wavefunction $\psi(x) = e^{ikx}$) is incident on a delta function potential at the origin,

$$V(x) = v_0\, \delta(x) \ . \tag{1}$$

a) (*2 pts*) By integrating the 1D Schödinger equation from $x = -\epsilon$ to $x = \epsilon$, show that this delta function potential leads to a discontinuity in ψ' of the form

$$\lim_{\epsilon \to 0} \left\{ \psi'(x = \epsilon) - \psi'(x = -\epsilon) \right\} = +\frac{2m}{\hbar^2}\, v_0\, \psi(x = 0) \ . \tag{2}$$

Assume in the following that the full solution of the 1D Schrödinger equation consists of the incident wave e^{ikx}, a reflected wave with amplitude r, and a transmitted wave with amplitude t,

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx}, & x < 0 \\ te^{ikx}, & 0 < x \ . \end{cases} \tag{3}$$

b) (*2 pts*) Impose this $\psi'(x)$ boundary condition and $\psi(x)$ continuity at $x = 0$, and show that this gives the constraints

$$ikt - ik(1 - r) = +\frac{2mv_0}{\hbar^2} t \tag{4}$$

and

$$1 + r = t \ . \tag{5}$$

c) (*2 pts*) Solve for r and t. They should be expressed as simple functions of the scaled momentum variable $\kappa = k\hbar^2/mv_0$.

d) (*2 pts*) Find the transmission and reflection probilities $P_t = |t|^2$ and $P_r = |r|^2$, and plot them versus κ over the range $\kappa = [0, 10]$. Check that $|t|^2 + |r|^2 = 1$.

e) (*2 pts*) The transmitted wave phase shift δ_t is defined by $t = |t|\, e^{i\delta_t}$. Show that it satisfies

$$\tan(\delta_t) = -1/\kappa \ , \tag{6}$$

and plot this δ_t as above. (You may assume $v_0 > 0$; this and $\lim_{k\to\infty} \delta_t = 0$ implies which branch of arctangent to use.)

2. Scattering and the "delta-shell" potential.

The "delta-shell" potential (in its 3D form c/o J.Mandula, c/o K.Gottfried) is a rather more exciting barrier potential which can be used to study resonance effects. In this 1D problem an interior region is separated from the exterior by a delta-function shell;

$$V(x) = v_0 \Big(\delta(x) + \delta(x - R)\Big) \ . \tag{7}$$

At $v_0 = \infty$ we have stable bound states in the interior region and total reflection of any incident waves on the outside. As we decrease v_0 the interior bound states will couple with increasing strength to the continuum of external scattering states, so we expect them to develop widths and shift their (suitably defined) energies.

To study scattering from this potential we again assume a unit strength incident wave from the left and reflected and transmitted waves going to $x = -\infty$ and $x = +\infty$ respectively. Since $V = 0$ except at $x = 0$ and $x = R$, the wavefunction consists of free plane waves,

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx}, & x < 0 \\ ce^{ikx} + de^{-ikx}, & 0 < x < R \\ te^{ikx}, & R < x \ . \end{cases} \tag{8}$$

a) (*2 pts*) Show that imposing boundary conditions (analogous to problem 1) at $x = 0$ and $x = R$ gives four constraints on the four unknowns r, c, d, t,

$$ik(c - d) - ik(1 - r) = u_0(1 + r) \tag{9}$$

$$c + d = 1 + r \tag{10}$$

$$ikte^{ikR} - ik(ce^{ikR} - de^{-ikR}) = u_0 te^{ikR} \tag{11}$$

$$te^{ikR} = ce^{ikR} + de^{-ikR} \tag{12}$$

where we have abbreviated $2mv_0/\hbar^2 = u_0$.

b) (*4 pts*) Solve the first two constraint equations Eqs.(9,10) for c and d, and use these results to eliminate c and d in the second two equations, Eqs(11,12). You may then eliminate r, leaving the final result for t. Hopefully you will find

$$t = \left\{ 1 + i\frac{u}{kR} + \frac{u^2}{4(kR)^2}\left(e^{2ikR} - 1\right) \right\}^{-1} \tag{13}$$

where $u = u_0 R = 2mv_0 R/\hbar^2$.

c) (*2 pts*) Given Eq.(13), check that as the two delta functions are superimposed ($R \to 0$, v_0 fixed) you recover a transmission amplitude equivalent to the t you found in problem 1.c.

d) (*2 pts*) The transmission probability $P_t = |t|^2$ (introducing $\chi = kR$) is

$$P_t = \left[1 + \frac{u^2}{2\chi^2} + \frac{u^4}{8\chi^4} + \left(\frac{u^2}{2\chi^2} - \frac{u^4}{8\chi^4}\right)\cos(2\chi) + \frac{u^3}{2\chi^3}\sin(2\chi)\right]^{-1}. \tag{14}$$

Plot this function over the range $\chi = [0, 20]$ (with y-axis $P_t = [0, 1]$) for $u = 1, 10$ and 100; note and explain qualitatively the physical reason for the changing peak positions and widths.

3. Transmission through a 1D barrier.

In class we showed that the transmission probability through a 1D potential barrier of the form

$$V(x) = \begin{cases} 0, & x < 0 \text{ or } x > R \\ V_0 > 0, & R > x > 0 \end{cases} \tag{15}$$

is given by

$$P_t = \left[1 + \frac{1}{4\epsilon(\epsilon - 1)}\sin^2(\xi\sqrt{\epsilon - 1})\right]^{-1} \tag{16}$$

for $E > V_0$ (where $\epsilon = E/V_0$ and $\xi = (2mV_0)^{1/2}R/\hbar$).

a) (*4 pts*) Find the corresponding transmission probability P_t for an *attractive* barrier, $V_0 = -|V_0| < 0$, by simply replacing the two occurrences of $(\epsilon - 1)$ by $(\epsilon + 1)$, and generalizing the dimensionless quantities to $\epsilon = E/|V_0|$ and $\xi = (2m|V_0|)^{1/2}R/\hbar$.

b) (*4 pts*) Plot this new transmission probability over the range $\epsilon = [0, 10]$ for $\xi = 1$ and $\xi = 10$. Do the maxima correspond to $\lambda = (2/n)R$ inside the well as in the repulsive case?

c) (*2 pts*) What is P_t in the low-energy limit ($\epsilon \to 0$)?