Clebsch - Gordon problem

Generally when we combine spin S_1 and S_2 states we obtain a series of S_{tot} multiplets:

$$\boxed{S_1 \otimes S_2 = \sum_{S_{tot}} \oplus S_{tot}}$$

e.g.

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

obviously the max S_{tot} is S_1+S_2 in general, since the largest $S_{z\,tot}$ comes from the stretched state;

$$\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2$$

$$S_{z\,tot} = S_{z1} + S_{z2}$$

max $S_{z_1} = S_1$, $S_{z_2} = S_2$ "stretched state"

so max $S_{z\,tot} = S_1 + S_2$. This means the highest $S_{tot} = S_1 + S_2$.
Just one S_1+S_2 state is present, since the $|S_1, \underbrace{S_1}_{S_{1z}}\rangle\, |S_2, \underbrace{S_2}_{S_{2z}}\rangle$ state is unique.
Additional states are present on the rhs if $S_1, S_2 \neq 0$, as is clear from state counting:

$$\underbrace{(2S_1+1)\cdot(2S_2+1)}_{4S_1S_2 + 2(S_1+S_2)+1} > \underbrace{(2S_1+2S_2+1)}_{\text{states in } 2S_{tot}+1 \text{ multiplet with } S_{tot}=S_1+S_2}$$

$\underbrace{4S_1S_2}_{\text{as yet unaccounted for.}}$

Going down in $S_{z\,tot}$, what states are present?

$S_{z\,tot}$

s_1+s_2 —	$\|s_1,s_1\rangle\|s_2,s_2\rangle$	1 state
s_1+s_2-1 —	$\underbrace{\|s_1,s_1-1\rangle}_{\text{"}-1\text{"}}\underbrace{\|s_2,s_2\rangle}_{\text{"}0\text{"}}$, $\underbrace{\|s_1,s_1\rangle}_{\text{"}0\text{"}}\underbrace{\|s_2,s_2-1\rangle}_{\text{"}-1\text{"}}$	2 states
s_1+s_2-2 —	$\|-2,0\rangle$, $\|-1,-1\rangle$, $\|0,-2\rangle$	3 states
s_1+s_2-3 —	$\|-3,0\rangle$, $\|-2,-1\rangle$, $\|-1,-2\rangle$, $\|0,-3\rangle$	4 states
	note we are just counting	
s_1+s_2-m —	$\underbrace{\|-m\rangle, \|-m+1,-1\rangle, \|-m+2,-2\rangle, \ldots, \|0,-m\rangle}_{m+1 \text{ states}}$	m+1 states

$\uparrow S_{z\,tot}$

product states $|s_1 s_{1z}\rangle|s_2 s_{2z}\rangle$

(assume s_1 & s_2 are large so no truncation problem yet)

$\therefore$ Since each S_{tot} multiplet contributes 1 $S_{z\,tot}$ state, evidently we have just 1 $S_{tot}=s_1+s_2$, 1 $S_{tot}=s_1+s_2-1$, 1 $S_{tot}=s_1+s_2-2$ ...

How far does the S_{tot} series run? Determine this by counting

$$s_1 \otimes s_2 = \sum_{S_{tot}=S_{min}}^{S_{max}=s_1+s_2} \oplus S_{tot} \quad \leftarrow \text{one each}$$

$$(2s_1+1)(2s_2+1) = \sum_{\sigma=S_{min}}^{S_{max}=s_1+s_2} (2\sigma+1)$$

$$= S_{max}-S_{min}+1 + 2\sum_{\sigma=S_{min}}^{S_{max}} \sigma$$

$$\sum_{\sigma=1}^{n} \sigma = \frac{1}{2} n(n+1)$$

$$(2s_1+1)(2s_2+1) = s_1+s_2 - S_{min} + 1 + 2\left\{ \sum_{\sigma=1}^{S_{max}} \sigma - \sum_{\sigma=1}^{S_{min}-1} \sigma \right\}$$

$$= \underbrace{s_1+s_2 - S_{min} + 1}_{s_1+s_2-\cancel{S_{min}}+1} + \underbrace{2\left\{ \frac{1}{2}(s_1+s_2)(s_1+s_2+1) - \frac{1}{2}(S_{min}-1)S_{min} \right\}}_{(s_1+s_2)^2 + s_1+s_2 - S_{min}^2 + \cancel{S_{min}}}$$

$$4s_1 s_2 + \cancel{2(s_1+s_2)} + \cancel{1} = \cancel{1} + s_1^2 + s_2^2 + 2s_1 s_2 + \cancel{2(s_1+s_2)} - S_{min}^2$$

$$S_{min}^2 = (s_1 - s_2)^2$$

$$S_{min} = |s_1 - s_2| \quad \text{since it's } \geq 0 .$$

$$\therefore \quad s_1 \otimes s_2 = \underbrace{\sum_{S_{tot}=|s_1-s_2|}^{s_1+s_2} \oplus\, S_{tot}}_{\text{one state of each spin present.}}$$

$\boxed{\text{Can visualize this as} \quad \vec{S}_{tot} = \underbrace{\uparrow s_2 \, \uparrow s_1 \quad \cdots \quad s_1 \uparrow\downarrow s_2}_{\text{steps of } 1}}$

e.g.

$$\frac{1}{2} \otimes \frac{1}{2} = \left(\frac{1}{2} - \frac{1}{2}\right) \oplus \left(\frac{1}{2} + \frac{1}{2}\right) = 0 \oplus 1$$

$$\underset{\sim}{2} \cdot \underset{\sim}{2} = \underset{\sim}{1} + \underset{\sim}{3}$$

$$1 \otimes \frac{1}{2} = \left(1 - \frac{1}{2}\right) \oplus \left(1 + \frac{1}{2}\right) = \frac{1}{2} \oplus \frac{3}{2}$$

$$\underset{\sim}{3} \cdot \underset{\sim}{2} = \underset{\sim}{2} + \underset{\sim}{4}$$

$$s_1 \otimes 1/2 = \left(s_1 - \frac{1}{2}\right) \oplus \left(s_1 + \frac{1}{2}\right)$$

$$\underset{\sim}{(2s_1+1)} \cdot \underset{\sim}{2} = \underset{\sim}{2s_1} + \underset{\sim}{2s_1+2}$$

$$\tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2} = (0\oplus 1)\otimes\tfrac{1}{2} = 0\otimes\tfrac{1}{2} \oplus 1\otimes\tfrac{1}{2}$$

$$= \tfrac{1}{2} \oplus \tfrac{1}{2} \oplus \tfrac{3}{2}$$

$$\equiv \left(\tfrac{1}{2}\right)^2 \oplus \tfrac{3}{2}$$

$$\underset{\sim}{2}\cdot\underset{\sim}{2}\cdot\underset{\sim}{2} \overset{?}{=} \underset{\sim}{2} + \underset{\sim}{2} + \underset{\sim}{4} \checkmark$$

The expansion coefficients for these $|S_{tot}, M_{tot}\rangle \equiv |S\, M\rangle$ states are the general CG coeffs:

$$|S\,M\rangle = \sum_{m_1, m_2} \underbrace{\langle s_1\, m_1, s_2\, m_2 \,|\, S\, M\rangle}_{\text{a CG coeff.}} \, |s_1 m_1\rangle |s_2 m_2\rangle$$

(vanishes unless $m_1 + m_2 = M$ and $S \in s_1 \otimes s_2$)

examples we know already are from $\tfrac{1}{2}\ \tfrac{1}{2}$:

$$|1, 1\rangle = 1\cdot|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |++\rangle$$

$1 = \langle \tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2} \,|\, 1\ 1\rangle$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

$$\equiv \underbrace{\langle \tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ -\tfrac{1}{2} \,|\, 1\, 0\rangle}_{+\frac{1}{\sqrt{2}}} |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \underbrace{\langle \tfrac{1}{2}\ -\tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2} \,|\, 1\, 0\rangle}_{+\frac{1}{\sqrt{2}}} |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

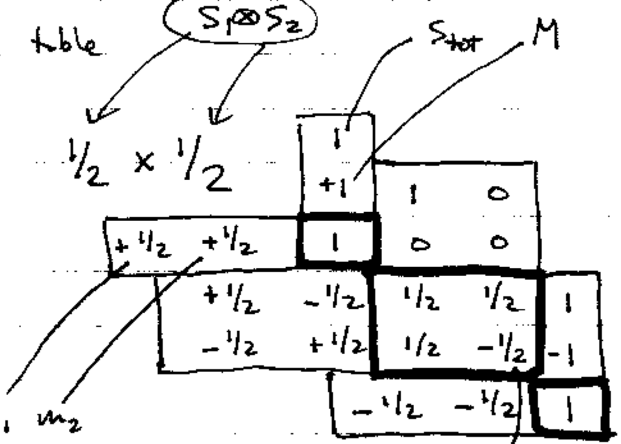
$$|1,-1\rangle = \underbrace{1}_{\langle \frac12\, -\frac12,\ \frac12\, -\frac12 \,|\, 1\, -1\rangle} \cdot |\tfrac12\, -\tfrac12\rangle|\tfrac12\, -\tfrac12\rangle = |--\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt2}\left(|+-\rangle - |-+\rangle\right)$$

$$= \underbrace{\langle \tfrac12\, \tfrac12,\ \tfrac12\, -\tfrac12 \,|\, 0 0\rangle}_{+\frac{1}{\sqrt2}} |\tfrac12\, \tfrac12\rangle|\tfrac12\, -\tfrac12\rangle$$

$$+ \underbrace{\langle \tfrac12\, -\tfrac12,\ \tfrac12\, \tfrac12 \,|\, 0 0\rangle}_{-\frac{1}{\sqrt2}} |\tfrac12\, -\tfrac12\rangle|\tfrac12\, \tfrac12\rangle$$

All this is summarized by the PDG table



suppress $\sqrt{\ }$ in tables of coeffs,

e.g. "$-\tfrac12$" $\equiv -\sqrt{\tfrac12}$

"$\tfrac12$" $= +\sqrt{\tfrac12}$

note $\displaystyle\sum_{m_1 m_2} \langle s_1 m_1, s_2 m_2 | S M\rangle^2 = 1,$

since that's the $\sum$ of coeffs^2 in the expansion of a normalized state.

Note each $\boxed{\text{CGs}}$ block is a unitary matrix, since they are just a change of orthonormal basis.

Isn't this obvious? Why bother with tabulation?

Not obvious what these are for higher spin.
Sum formulas $\exists$ but they are quite complicated.
More useful to tabulate the lowest few $S_1 \otimes S_2$ cases explicitly.

Alternative (inverse) expansion.
We know e.g.

$$|+-\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$$

i.e. we can expand an $|s_1 m_1, s_2 m_2\rangle$ state in a series of states of definite S_{tot}, just the inverse of the CG series

$$|s_1 m_1\rangle|s_2 m_2\rangle = \sum_S \underbrace{\langle S\ M=m_1+m_2 | s_1 m_1, s_2 m_2\rangle}_{\text{CG coeffs again}} |S\ M\rangle$$

e.g.

$$|+-\rangle = |\tfrac12\ {+\tfrac12}\rangle|\tfrac12\ {-\tfrac12}\rangle = \sum_{S=0}^{1} \langle S\,0 | \tfrac12\,\tfrac12, \tfrac12\,{-\tfrac12}\rangle\, |S\,0\rangle$$

$$= \underbrace{\langle 1 0 | \tfrac12\,\tfrac12, \tfrac12\,{-\tfrac12}\rangle}_{+1/\sqrt{2}} |1 0\rangle$$

$$+ \underbrace{\langle 0 0 | \tfrac12\,\tfrac12, \tfrac12\,{-\tfrac12}\rangle}_{+1/\sqrt{2}} |0 0\rangle$$

$$= +\frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$$ (horiz. row in CG table)

since this init state has unit norm,

$$\sum_S \langle S\ M=m_1+m_2 | s_1 m_1, s_2 m_2\rangle^2 = 1$$

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	1
		-1
-1/2	-1/2	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	1/2	1

m_1	m_2	5/2	3/2
		3/2	+3/2
+2	-1/2	1/5	4/5
+1	+1/2	4/5	-1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	-1/2	2/5	3/5
0	+1/2	3/5	-2/5

m_1	m_2	5/2	3/2
		-1/2	-1/2
0	-1/2	3/5	2/5
-1	+1/2	2/5	-3/5

m_1	m_2	5/2	3/2
		-3/2	-3/2
-1	-1/2	4/5	1/5
-2	+1/2	1/5	-4/5

m_1	m_2	5/2
		-5/2
-2	-1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	-1/2	1/3	2/3
0	+1/2	2/3	-1/3

m_1	m_2	3/2	1/2
		-1/2	-1/2
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3

m_1	m_2	3/2
		-3/2
-1	-1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	-1/2	1/4	3/4
+1/2	+1/2	3/4	-1/4

m_1	m_2	2	1
		0	0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	2	1
		-1	-1
-1/2	-1/2	3/4	1/4
-3/2	+1/2	1/4	-3/4

m_1	m_2	2
		-2
-3/2	-1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	-1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	-1	1/15	1/3	3/5
+1	0	8/15	1/6	-3/10
0	+1	6/15	-1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	-1	1/5	1/2	3/10
0	0	3/5	0	-2/5
-1	+1	1/5	-1/2	3/10

m_1	m_2	3	2	1
		-1	-1	-1
0	-1	6/15	1/2	1/10
-1	0	8/15	-1/6	-3/10
-2	+1	1/15	-1/3	3/5

m_1	m_2	3	2
		-2	-2
-1	-1	2/3	1/3
-2	0	1/3	-2/3

m_1	m_2	3
		-3
-2	-1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	-1/2

m_1	m_2	2	1	0
		0	0	0
+1	-1	1/6	1/2	1/3
0	0	2/3	0	-1/3
-1	+1	1/6	-1/2	1/3

m_1	m_2	2	1
		-1	-1
0	-1	1/2	1/2
-1	0	1/2	-1/2

m_1	m_2	2
		-2
-1	-1	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	-2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	-1	1/10	2/5	1/2
+1/2	0	3/5	1/15	-1/3
-1/2	+1	3/10	-8/15	1/6

m_1	m_2	5/2	3/2	1/2
		-1/2	-1/2	-1/2
+1/2	-1	3/10	8/15	1/6
-1/2	0	3/5	-1/15	-1/3
-3/2	+1	1/10	-2/5	1/2

m_1	m_2	5/2	3/2
		-3/2	-3/2
-1/2	-1	3/5	2/5
-3/2	0	2/5	-3/5

m_1	m_2	5/2
		-5/2
-3/2	-1	1

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	-1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	-1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	-2/5
-1/2	+3/2	1/5	-1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	-3/2	1/20	1/4	9/20	1/4
+1/2	-1/2	9/20	1/4	-1/20	-1/4
-1/2	+1/2	9/20	-1/4	-1/20	1/4
-3/2	+3/2	1/20	-1/4	9/20	-1/4

m_1	m_2	3	2	1
		-1	-1	-1
+1/2	-3/2	1/5	1/2	3/10
-1/2	-1/2	3/5	0	-2/5
-3/2	+1/2	1/5	-1/2	3/10

m_1	m_2	3	2
		-2	-2
-1/2	-3/2	1/2	1/2
-3/2	-1/2	1/2	-1/2

m_1	m_2	3
		-3
-3/2	-3/2	1

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	-1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	-2/5
0	+3/2	2/7	-18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	-3/2	1/35	6/35	2/5	2/5
+1	-1/2	12/35	5/14	0	-3/10
0	+1/2	18/35	-3/35	-1/5	1/5
-1	+3/2	4/35	-27/70	2/5	-1/10

m_1	m_2	7/2	5/2	3/2	1/2
		-1/2	-1/2	-1/2	-1/2
+1	-3/2	4/35	27/70	2/5	1/10
0	-1/2	18/35	3/35	-1/5	-1/5
-1	+1/2	12/35	-5/14	0	3/10
-2	+3/2	1/35	-6/35	2/5	-2/5

m_1	m_2	7/2	5/2	3/2
		-3/2	-3/2	-3/2
0	-3/2	2/7	18/35	1/5
-1	-1/2	4/7	-1/35	-2/5
-2	+1/2	1/7	-16/35	2/5

m_1	m_2	7/2	5/2
		-5/2	-5/2
-1	-3/2	4/7	3/7
-2	-1/2	3/7	-4/7

m_1	m_2	7/2
		-7/2
-2	-3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	-1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	-3/7
0	+2	3/14	-1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	-1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	-1/14	-3/10
0	+1	3/7	-1/5	-1/14	3/10
-1	+2	1/14	-3/10	3/7	-1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	-2	1/70	1/10	2/7	2/5	1/5
+1	-1	8/35	2/5	1/14	-1/10	-1/5
0	0	18/35	0	-2/7	0	1/5
-1	+1	8/35	-2/5	1/14	1/10	-1/5
-2	+2	1/70	-1/10	2/7	-2/5	1/5

m_1	m_2	4	3	2	1
		-1	-1	-1	-1
+1	-2	1/14	3/10	3/7	1/5
0	-1	3/7	1/5	-1/14	-3/10
-1	0	3/7	-1/5	-1/14	3/10
-2	+1	1/14	-3/10	3/7	-1/5

m_1	m_2	4	3	2
		-2	-2	-2
0	-2	3/14	1/2	2/7
-1	-1	4/7	0	-3/7
-2	0	3/14	-1/2	2/7

m_1	m_2	4	3
		-3	-3
-1	-2	1/2	1/2
-2	-1	1/2	-1/2

m_1	m_2	4
		-4
-2	-2	1