an aside on
SHO matrix elements

How to calculate $\langle n|x^2|n\rangle$ using explicit wfns.

$$\psi_n(x) = \eta_n H_n(s)\, e^{-\frac{1}{2}s^2} \qquad \eta_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{c}{\pi}\right)^{1/4}, \quad c = \frac{(km)^{1/2}}{\hbar}$$

$$s = \sqrt{c}\,x$$

$$\langle n|x|n\rangle = \eta_n^2 \int_{-\infty}^{\infty} H_n(s) e^{-\frac{1}{2}s^2} \underbrace{x H_n(s)}_{\to \frac{1}{\sqrt{c}} s H_n(s)} e^{-\frac{1}{2}s^2}\, dx$$

$$s H_n(s) = \frac{1}{2} H_{n+1}(s) + n H_{n-1}(s)$$

$$= \frac{\eta_n^2}{\sqrt{c}} \int_{-\infty}^{\infty} H_n(s)\left\{\frac{1}{2}H_{n+1}(s) + n H_{n-1}(s)\right\} e^{-s^2}\, dx$$

$$= 0$$

similarly

$$\langle n|x^2|n\rangle = \eta_n^2 \int_{-\infty}^{\infty} H_n(s) e^{-\frac{1}{2}s^2} \underbrace{x^2 H_n(s)}_{\frac{1}{c}s^2 H_n(s)} e^{-\frac{1}{2}s^2} \underbrace{dx}_{\frac{1}{\sqrt{c}}ds}$$

$$s^2 H_n(s) = \frac{1}{2} s H_{n+1} + n s H_{n-1} = \frac{1}{2}\left\{\frac{1}{2}H_{n+2} + (n+1) H_n\right\} + n\left\{\frac{1}{2} H_n + (n-1) H_{n-2}\right\}$$

$$= \frac{\eta_n^2}{c} \int_{-\infty}^{\infty} H_n \cdot \left\{\frac{1}{4}\cancel{H_{n+2}} + (n+\tfrac{1}{2}) H_n + n(n-1)\cancel{H_{n-2}}\right\} e^{-s^2}\, dx$$

$\int \to 0$ $\qquad\qquad$ $\int \to 0$

$$= \frac{1}{c}(n+\tfrac{1}{2}) = \underline{\frac{\hbar}{(km)^{1/2}}(n+\tfrac{1}{2})}$$

n.b. $\langle n|\frac{1}{2}kx^2|n\rangle = \frac{1}{2}\sqrt{\frac{k}{m}}\,\hbar(n+\tfrac{1}{2})$

$= \frac{1}{2}(n+\tfrac{1}{2})\hbar\omega = \frac{1}{2}E_n$ ✓

A neater way of writing this:

$$x\psi_n = \frac{\eta_n}{\sqrt{c}}\; s H_n(s)\, e^{-\frac{1}{2}s^2}$$

$$= \frac{1}{\sqrt{c}}\,\eta_n \left\{ \tfrac{1}{2} H_{n+1} + n H_{n-1} \right\} e^{-\frac{1}{2}s^2}$$

$$= \frac{1}{\sqrt{c}} \left\{ \frac{1}{2} \underbrace{\frac{\eta_n}{\eta_{n+1}}}_{\sqrt{2(n+1)}} \underbrace{\eta_{n+1} H_{n+1} e^{-\frac{1}{2}s^2}}_{\psi_{n+1}} + n \underbrace{\frac{\eta_n}{\eta_{n-1}}}_{\frac{1}{\sqrt{2n}}} \underbrace{\eta_{n-1} H_{n-1} e^{-\frac{1}{2}s^2}}_{\psi_{n-1}} \right\}$$

or

$$x\psi_n = \frac{1}{\sqrt{2c}} \left\{ \sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\,\psi_{n-1} \right\}$$

or in Dirac notation

$$x|n\rangle = \frac{1}{\sqrt{2c}} \left\{ \sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle \right\}$$

then clearly

$$x^2|n\rangle = \frac{1}{2c} \left\{ \sqrt{(n+1)(n+2)}\,|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}\,|n-2\rangle \right\}$$

$$\langle n|x^2|n\rangle = \frac{1}{2c} \left\{ \sqrt{\quad}\,\underbrace{\langle n|n+2\rangle}_{0} + (2n+1)\underbrace{\langle n|n\rangle}_{1} + \sqrt{n(n-1)}\,\underbrace{\langle n|n-2\rangle}_{0} \right\}$$

$$= \frac{1}{c}\left(n+\frac{1}{2}\right) = \frac{\hbar}{(km)^{1/2}}\left(n+\frac{1}{2}\right) \checkmark$$

Similarly you can work out "momentum" expected values like

$$\langle n|p_x|n\rangle = -i\hbar\int_{-\infty}^{\infty}\psi_n^*\,\psi_n'\,dx$$

using $$\frac{d}{ds}H_n(s) = 2nH_{n-1}(s)$$

In Dirac's form,

$$p_x|n\rangle \to \underbrace{-i\hbar\frac{d}{dx}}_{\sqrt{c}\frac{d}{ds}}\,\eta_n H_n(s)\,e^{-\frac{1}{2}s^2}$$

$$= -i\hbar\eta_n\sqrt{c}\left(\frac{dH_n}{ds} - sH_n\right)e^{-\frac{1}{2}s^2}$$

$$= -i\hbar\sqrt{c}\,\eta_n\left\{2nH_{n-1}\underbrace{-sH_n}_{-\frac{1}{2}H_{n+1}-nH_{n-1}}\right\}e^{-\frac{1}{2}s^2}$$

$$= -i\hbar\sqrt{c}\,\eta_n\left\{-\frac{1}{2}H_{n+1} + nH_{n-1}\right\}e^{-\frac{1}{2}s^2}$$

$$= -i\hbar\sqrt{c}\left\{-\frac{1}{2}\underbrace{\frac{\eta_n}{\eta_{n+1}}}_{\sqrt{2(n+1)}}\psi_{n+1} + n\underbrace{\frac{\eta_n}{\eta_{n-1}}}_{\frac{1}{\sqrt{2n}}}\psi_{n-1}\right\}$$

$$= +\frac{i\hbar\sqrt{c}}{\sqrt{2}}\left\{+\sqrt{n+1}\,\psi_{n+1} - \sqrt{n}\,\psi_{n-1}\right\}$$

$$\therefore\quad p_x|n\rangle = i\hbar\sqrt{\frac{c}{2}}\left\{\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle\right\}$$ so $\langle n|p_x|n\rangle = 0$

$$p_x^2 |n\rangle = -\frac{\hbar^2 c}{2} \left\{ \sqrt{(n+1)(n+2)}\,|n+2\rangle - (2n+1)|n\rangle + \sqrt{n(n-1)}\,|n-2\rangle \right\}$$

$$\underbrace{\langle n | p_x^2 | n \rangle} = +\hbar^2 c \left(n+\tfrac{1}{2}\right) = \underbrace{\hbar \sqrt{km} \left(n+\tfrac{1}{2}\right)}$$

n.b. $\langle n | KE | n \rangle = \langle n | \frac{p_x^2}{2m} | n \rangle$

$$= \frac{1}{2m} \hbar \sqrt{km} \left(n+\tfrac{1}{2}\right)$$

$$= \frac{1}{2} \hbar \sqrt{\frac{k}{m}} \left(n+\tfrac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \underbrace{\left(n+\tfrac{1}{2}\right)\hbar\omega}_{E_n}$$