Two applications of CG coeffs.

1) Gaunt formula (product of two Y_{lm}s : most general QM angular m.e.)

2) Wigner - Eckart theorem (reduced matrix elements)

1) Gaunt formula

The most general angular matrix element we will generally encounter in QM is of the form

$$\int_0^\infty r^2 dr \underbrace{\int \psi^*_{l'm'} \, f(\Omega) \, \psi_{lm} \, d\Omega}_{\mathcal{J}_{ang}}$$

$\psi^*_{l'm'} = \varphi^*_{l'm'}(r) \, Y^*_{l'm'}(\Omega)$

$\psi_{lm} = \varphi_{lm}(r) \cdot Y_{lm}(\Omega)$

expanding f in spherical harmonics $f(\Omega) = \sum_{LM} c_{LM} Y_{LM}$
this reduces to

$$\mathcal{J}_{ang.} = \sum_{LM} c_{LM} \underbrace{\int d\Omega \, Y^*_{l'm'} \; Y_{LM} \; Y_{lm}}$$

the "Gaunt formula" is basically the result for this integral.

since $\int Y^*_{\lambda'\mu'} \, Y_{\lambda\mu} \, d\Omega = \delta_{\lambda\lambda'} \delta_{\mu\mu'}$, what we need is the expansion of a product of 2 spherical harmonics, $Y_{LM} Y_{lm}$, in spherical harmonics.

Since the $\{Y_{\lambda\mu}\}$ form a complete set this is an allowed expansion.

Note: since $Y_{lm}(\Omega)$ is the angular wfn for an $|lm\rangle$ eigenstate,
$$\langle \Omega | lm \rangle = Y_{lm}(\Omega),$$

it should not be surprising that the product

$$Y_{l_1 m_1} Y_{l_2 m_2} \quad \text{involves a CG series, similar to}$$

$$|l_1 m_1\rangle |l_2 m_2\rangle = \sum_{LM} \langle LM | l_1 m_1, l_2 m_2 \rangle \, |LM\rangle$$

(coefs are not only $\langle LM | l_1 m_1, l_2 m_2\rangle$: we are setting $\Omega_1 = \Omega_2$ in $Y_{l_1 m_1}(\Omega_1)\, Y_{l_2 m_2}(\Omega_2)$
so it's not exactly a two-angular-momentum problem)

<u>result:</u> (enlightening grunge)

forces $M = m_1 + m_2$, $l_1 + l_2 \geq L \geq |l_1 - l_2|$

$$Y_{l_1 m_1}(\Omega)\, Y_{l_2 m_2}(\Omega) = \sum_{LM} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)} \right]^{1/2} \langle LM | l_1 m_1, l_2 m_2 \rangle \cdot \underbrace{\langle L0 | l_1 0, l_2 0\rangle}_{} \cdot Y_{LM}(\Omega)$$

forces parity constraint $(-)^L = (-)^{l_1+l_2}$

(J.A. Gaunt, Trans. Roy. Soc. Lon. A 228, 195 (1928).)

e.g. (from linear Stark effect probs.)

$$\int Y_{l'm'}^* \cos\theta \, Y_{lm} \, d\Omega = \, ? \qquad \cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}$$

$$\sqrt{\frac{4\pi}{3}}\, Y_{10} Y_{lm} = \sum_{l'm'} \left[\frac{2l+1}{2l'+1} \right]^{1/2} \underbrace{\langle l'm' | 10, lm\rangle}_{\propto\, \delta_{mm'}} \underbrace{\langle l'0 | 10, l0 \rangle}_{\substack{l' = l \pm 1 \text{ only} \\ l' \neq l}} Y_{l'm'}$$

$$\cos\theta\, Y_{\ell m}(\Omega) = \left[\frac{2\ell+1}{2\ell+3}\right]^{1/2} \underbrace{\langle \ell+1\, m \mid 10,\, \ell m\rangle \cdot \langle \ell+1\, 0 \mid 10,\, \ell 0\rangle}_{\text{these are fairly simple } \int\text{'s.}} Y_{\ell+1,\, m}$$

$$+ \left[\frac{2\ell+1}{2\ell-1}\right]^{1/2} \underbrace{\langle \ell-1\, m \mid 10,\, \ell m\rangle \cdot \langle \ell-1,\, 0 \mid 1,0,\, \ell\, 0\rangle}_{\text{fairly simple } \int\text{'s.}} Y_{\ell-1,\, m}$$