PHY521; Quantum Mechanics, Problem Set 5

Due Wed. 27 Sep. 2006 at the beginning of class.

1. 3D Spherical Well

Consider a 3D (spherical) square well of radius a, in which the potential $V(r)$ is negative inside the well and zero outside,

$$V(x) = \begin{cases} -|V_0| < 0, & r < a \qquad \text{(inside)} \\ 0, & r > a \qquad \text{(outside).} \end{cases} \tag{1}$$

To find energy eigenstates we must solve the radial Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr}\right) + V(r)\,\psi(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\,\psi(r) = E\,\psi(r)\ . \tag{2}$$

a) (*5 pts*) First consider the S-waves (and recall the simplifying substitution $\psi(r) = u(r)/r$). Show that these states satisfy the constraint

$$-\chi\cot(\chi) = \sqrt{\frac{2m|V_0|a^2}{\hbar^2} - \chi^2} \tag{3}$$

where $\chi = ka$. Unlike the 1D case there is a minimum depth $|V_0|$ required to support a bound state. What is this $|V_0|$?

b) (*5 pts*) Find the corresponding constraint equation and minimum binding energy $|V_0|$ for P-wave states ($\ell = 1$). Give my regards to Herr Bessel.

2. Angular Momentum Algebra

Given the explicit orbital angular momentum operator's components in Cartesian coordinates,

$$\mathrm{L_x} = -\mathrm{i}\hbar\left(\mathrm{y}\,\frac{\partial}{\partial \mathrm{z}} - \mathrm{z}\,\frac{\partial}{\partial \mathrm{y}}\right) \tag{4}$$

et cycl., derive the following commutation relations:

a) (*2 pts*)

$$[\mathrm{L_x}, \mathrm{L_y}] = +\mathrm{i}\hbar\,\mathrm{L_z} \tag{5}$$

b) (*2 pts*)

$$[\vec{\mathrm{L}}^{\,2}, \mathrm{L_z}] = 0 \tag{6}$$

c) (*2 pts*)

$$[\mathrm{L_z}, \mathrm{L_+}] = +\hbar \mathrm{L_+} \tag{7}$$

where $\mathrm{L_+} = \mathrm{L_x} + \mathrm{i}\mathrm{L_y}$.

d) (*4 pts*) Use part c) to show that the effect of $\mathrm{L_+}$ on an $\mathrm{L}^2, \mathrm{L_z}$ eigenstate $|\ell, m\rangle$ is to increase the eigenvalue of $\mathrm{L_z}$ by $\hbar$.

3. Monsieur Legendre

Prove the following identities involving Legendre polynomials:

a) (*3 pts*)

$$P_{\ell+1}(\mu) = \frac{1}{(\ell+1)}\Big\{(2\ell+1)\,\mu\,P_\ell(\mu) - \ell\,P_{\ell-1}(\mu)\Big\}\ . \tag{8}$$

b) (*3 pts*)

$$(\mu^2-1)\,\frac{dP_\ell(\mu)}{d\mu} = \ell\mu\,P_\ell(\mu) - \ell\,P_{\ell-1}(\mu) \tag{9}$$

c) (*4 pts*)

$$\int_{-1}^{1} d\mu\,P_\ell(\mu)\,P_{\ell'}(\mu) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\ . \tag{10}$$

You may use as your starting points the generating function

$$f(\mu,x) = \Big(1-2x\mu+x^2\Big)^{-1/2} = \sum_{\ell=0}^{\infty} x^\ell\,P_\ell(\mu) \tag{11}$$

and Rodrigues' formula

$$P_\ell(\mu) = \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{d\mu^\ell}\Big(\mu^2-1\Big)^\ell\ . \tag{12}$$