## PHY521; Quantum Mechanics, Problem Set 1

Due Wednesday 30 Aug 2006 at the beginning of class.

## 1. Classical Hydrogen MSDS.

Since an accelerated charge in classical mechanics radiates energy at a rate of

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^2 a^2}{c^3} \ , \tag{1}$$

the classical H atom is unstable, and the electron will spiral into the proton.

a) (*5 pts*) Show from the above equation (assuming quasicircular motion) that the radius of the orbit decreases with time as

$$\frac{dr}{dt} = -\frac{4}{3}\frac{e^4}{m^2 c^3}\frac{1}{r^2} \ . \tag{2}$$

Integrate this equation and find the time $t_0$ it takes the electron to fall from an initial $r_0$ to $r = 0$. For $r_0 = \hbar^2/me^2 \approx 0.529$ Å(the Bohr radius), what is this lifetime in seconds?

b) (*5 pts*) What power (in watts) would 1 [gm] of this "classical hydrogen" radiate initially, if each electron starts at one Bohr radius from the proton?

## 2. Planck's Black Body Radiation Formula.

The formula derived by Planck for the differential (in wavelength) power density emitted by a black body is

$$P(\lambda) = \frac{c}{4} \cdot \frac{8\pi}{\lambda^5} \frac{hc}{\left(e^{hc/k\lambda T} - 1\right)} \tag{3}$$

where $c$ is the speed of light, $h$ is Planck's constant, $k$ is Boltzmann's constant, and $T$ is the temperature. (The factor of $c/4$ is needed to convert the differential energy density per wavelength to the differential radiated power per unit area per wavelength.)

a) (*2 pts*) Given the definition of $P(\lambda)$, which is that the differential power density $d\mathbf{P}$ (energy radiated per unit area per second) radiated from a black body between wavelengths $\lambda$ and $\lambda + d\lambda$ is

$$d\mathbf{P} = P(\lambda)\, d\lambda, \tag{4}$$

what (cgs) units should $P(\lambda)$ have, using ergs, cm and sec? Confirm that the $P(\lambda)$ expression quoted above does indeed have these units.

b) (*3 pts*) The wavelength $\lambda_{max}$ at which $P(\lambda)$ is a maximum is a pure number $c_0$ times a constant with the units of length,

$$\lambda_{max} = c_0\, \frac{hc}{kT} \tag{5}$$

Write a program to evaluate $P(\lambda)$ (please attach this to your HW set), and use it find the value of $c_0$ to at least four significant figures.

c) (*5 pts*) The *total* power per unit area $\mathbf{P}$ radiated from a black body can be determined by integrating this $P(\lambda)$ over all $\lambda$. Change variables to the more sensible dimensionless one $x = hc/k\lambda T$, and carry out this integral explicitly. Your result for $\mathbf{P}$ should be a constant $\sigma$ times $T^4$; this formula is the "Stefan-Boltzmann law". Incidentally, you now have a numerical QM prediction for the Stefan-Boltzmann constant $\sigma$, which is experimentally (in mks units) $\approx 5.67 \cdot 10^{-8}$ [W m$^{-2}$ K$^{-4}$]. Although the $T^4$ dependence was first derived using classical thermodynamics, determining the overall constant requires quantum mechanics.

## 3. Bohr Hydrogen.

In the Bohr (old QM) model of hydrogen, the electron can most simply be assumed to move in a circular orbit about the proton, with a Hamiltonian of

$$H = \frac{mv^2}{2} - \frac{e^2}{r} \ . \tag{6}$$

The allowed orbits are determined by a postulated quantization of orbital angular momentum, $L = mvr = n\hbar$.

a) (*3 pts*) Using classical mechanics, show that the radius and velocity of the electron in the n$^{th}$ Bohr orbit are given by $r_n = n^2 a_0$ (where $a_0 = \hbar/mc\,\alpha$ is the Bohr radius) and by $v_n = (\alpha/n) \cdot c$.

b) (*2 pts*) Obtain expressions for the PE and KE versus $n$, and show that they satisfy the virial theorem.

c) (*3 pts*) In his 1922 paper, F.S.Brackett reported a transition in hydrogen at a wavelength of $\lambda = 2.63 \pm 0.2$ [$\mu$m]. Compare this to the wavelengths of photons produced in $n_i \rightarrow n_f$ transitions in the Bohr model of the H atom, in which the $n^{th}$ level has the energy

$$E_n = -\frac{mc^2\alpha^2}{2n^2} \ . \tag{7}$$

What are the smallest plausible values of $n_i$ and $n_f$ for this transition? (Evaluate $\lambda$ using this energy level formula and the numerical values of $m$, $c$ and $\hbar$.)

d) (*2 pts*) The Rydberg constant for hydrogen, according to the Bohr model, should be given by $R_H = mc\,\alpha^2/4\pi\hbar$. Officially it is $R_H = 109677.58$ [cm$^{-1}$]. Compare this number with the value you find from $mc\,\alpha^2/4\pi\hbar$, and explain any apparent discrepancy. (Use the NIST website to find the 2006 values of these constants; our class website has a link.)