Chapter 9
Profit Maximization

Economic theory normally uses the profit maximization assumption in studying the firm just as it uses the utility maximization assumption for the individual consumer. This approach is taken to satisfy the need for a simple objective for the firm. This objective seems to be the most feasible.
• The profit-maximizing firm chooses both inputs and outputs so as to maximize the difference between total revenue and total cost.

• \( \pi = R(q) - C(q) \)

• The firm will adjust variables under its control until it cannot increase profit further. Thus, the firm looks at each additional unit of input and output with respect to its effect on profit.
\[ R(q) = p(q) \cdot q \]  
\[ \pi(q) = p(q) \cdot q - C(q) = R(q) - C(q) \]

To Maximize \( \pi \):

\[ \frac{d\pi}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0 \Rightarrow \frac{dR}{dq} = \frac{dC}{dq} \text{ or } MR = MC \]

Maximizing \( \pi \) is different from maximizing quantity (q) subject to a cost constraint (C=?) or minimizing C subject to a quantity constraint (q = ?). Find q that maximizes \( \pi \) and \( \pi = f(q) \), so one variable and no constant.
• MR=MC is the profit maximization rule --- Marginalism (MR is the change in R resulting from a small change in output and MC is the change in C resulting from a small change in output.)

• The SOC for profit maximization is: \( \frac{d^2 \pi}{dq^2} \Big|_{q^*} < 0 \).

At the optimal quantity \((q^*)\), marginal profit must be declining; economic profit \([\pi(q)]\) must be a concave function of \(q\) at \(q^*\).
At $q^*$, the slope of $C$ equals the slope of $R$ \( \Rightarrow MC = MR \) and $\pi$ is at a maximum. But $MC = MR$ at two points; one is at maximum $\pi$ and the other is at minimum $\pi$. Must check the SOC at $q^*$.

$$\frac{d^2 \pi}{dq^2} \bigg|_{q^*} < 0.$$  

At the other $MC = MR$, the second derivative of $\pi$ is $> 0!$
Let’s examine marginal revenue only.

\[ \frac{dR}{dq} = MR(q) = \frac{d[p(q) \cdot q]}{dq} = p + q \frac{dp}{dq} \]

MR = p for a perfectly elastic D curve, \( \frac{dq}{dp} = -\infty \), so e_{q,p} = -\infty

MR < p for a downward sloping D curve, which happens when more output can be sold only if the price is reduced for all units sold.
If \( \frac{dp}{dq} = 0 \) then \( MR = p \).

If \( \frac{dp}{dq} < 0 \) (downward sloping demand curve), then \( MR < p \).

\( MR \) is a function of \( q \) if \( \frac{dp}{dq} < 0 \).

**REMEMBER:** \( MR(q) = p + q \frac{dp}{dq} \) and 

\[ e_{qp} = \frac{dq}{dp} \cdot \frac{p}{q} \] (from a firm’s demand curve perspective); therefore,

\[ MR = p(1 + \frac{q}{p} \cdot \frac{dp}{dq}) = p(1 + \frac{1}{e_{qp}}) \]

This formula is derived by multiplying the second part of \( MR(q) \) by \( p/p \) and factoring out \( p \).
Given $\text{MR} = p(1 + \frac{1}{e_{q,p}})$;

with a negatively sloped demand curve, $e_{q,p}$ is negative and $p$ is greater than $\text{MR}$. Furthermore, if $e_{q,p} = -\infty$, $\text{MR} = p$. In summary:

If: $e_{q,p} = -\infty$, $\text{MR} = p$.
- $-\infty < e_{q,p} < 0$, $\text{MR} < p$.

If: $e_{q,p} < -1$ (demand is elastic), $\text{MR} > 0$.
- $e_{q,p} = -1$ (demand is unit elastic), $\text{MR} = 0$.
- $e_{q,p} > -1$ (demand is inelastic), $\text{MR} < 0$.

See example for a linear demand curve on the next slide.
If $R = 100q - q^2$, then $AR = p = 100 - q$ and $MR = 100 - 2q$.

$AR$ can be derived from chords to the $R$ curve.

$e_{q,p} < -1$ (elastic) \[ e_{q,p} = -1 \] (inelastic)

$MR > 0$ \[ MR < 0 \]

$p = AR = \$100$

$p = AR = \$50$

When $AR$ is declining, $MR$ is below it. For a linear demand curve, the slope of $MR$ is twice the slope of $AR$ in absolute value.

The firm’s demand curve is the firm’s $AR$ curve if the firm must sell all its output at one price.
Inverse Elasticity Rule

Given that MC = MR at maximum π for the firm and

\[ MR = p(1 + \frac{1}{e_{q,p}}) \]  then

\[ MC = p(1 + \frac{1}{e_{q,p}}) \]

\[ MC = p + \frac{p}{e_{q,p}} \]

\[ p - MC = -\frac{p}{e_{q,p}} \]

\[ \frac{p - MC}{p} = -\frac{1}{e_{q,p}} \]

This rule only makes sense if MR ≥ 0 because if MR < 0, MC < 0 at that q also, and MC < 0 is not possible. Therefore, a profit-maximizing firm will only operate in the elastic portion of its demand curve where MR > 0. This statement does not apply to industry demand curves.
Given \[ \frac{p - MC}{p} = -\frac{1}{e_{q,p}}, \]

- As \( e_{q,p} \) becomes more negative, \( \frac{p - MC}{p} \) becomes smaller, i.e., the gap between \( p \) and \( MC \) (\( p - MC \)) becomes smaller.

- When \( e_{q,p} = -\infty \), \( MC = p = MR \) at point of \( \pi \) Max.

- When the demand curve is negatively sloped, \( MR \) is below the demand curve (AR curve) and \( p \) is greater than \( MC \) at the quantity where \( MC = MR \).

Economist can look at the inverse elasticity to tell how close \( p \) is to \( MC \). As \( p \) approaches \( MC \), demand becomes more elastic. The inverse elasticity is a measure of market power.

At \( q^* \), \( p_1 - MC_1 > 0 \Rightarrow \) demand is downward sloping.
Profit Maximization by Price-Taking Firm

The firm is a price taker in the short run.

Economic $\pi$ is the area of the rectangle $= (p_1 - SAC_1)q^*.$

- At $q > q^*$, SMC > SMR so $\pi$ ↑ as $q \downarrow$.
- At $q < q^*$, SMC < SMR so $\pi$ ↑ as $q \uparrow$.
- Profit ↑ up to $q^*$ and falls beyond $q^*$.

$\pi(q) = p \cdot q - SC(q)$ \Rightarrow FOC: $\frac{d\pi}{dq} = \frac{dR}{dq} - \frac{dSC}{dq} = 0$

$\pi(q) = p_1 \cdot q^* - SAC_1$  
$\pi(q) = p_1 - SAC_1.$

FOC: $\frac{d\pi}{dq} = p - \frac{dSC}{dq} = 0$

SMC at $q^*$ must be increasing.
If: $\pi(q) = pq - SC(q)$
FOC: $\pi'(q) = p - SMC(q) = 0$
SOC: $\pi''(q) = - SMC'(q) < 0$
because $p' = 0$ for price-taker.
True only if $SMC'(q) > 0.$
**Price-taking Firm’s Short-Run Supply Curve**

- Because SMC shows how much the firm will produce at each price, it is the firm’s short-run supply curve. Set SMC = p and solve for q to get short-run supply function.

- The firm will move up and down the curve so SMR = SMC, maximizing π.

- At prices below $p_0$ the firm will produce zero output because it cannot cover SAVC. The firm will minimize losses by shutting down completely and only losing SAFC. If it continues to operate, it will lose all of SAFC and part of SAVC.

- At prices between $p_0$ and $p_2$, the firm will minimize losses (max $\pi$) by continuing to operate to cover all SAVC and part of SAFC. It loses all SAFC if it shuts down, but only part of SAFC if it operates.

- At prices above $p_2$ the firm earns an economic profit.

- Thus, the short-run supply curve is SMC above the minimum level of SAVC curve. SMC must be positively sloped also (SOC).
Profit Functions

Economic profit is defined as \( \pi = pf(K, L) - vK - wL \). This is not the Profit Function. The Profit Function is \( \pi^* = pf(K^*, L^*) - vK^* - wL^* \), or as in the text, \( \Pi(p, v, w) = \max_{K, L} \pi(K, L) = \max_{K, L} [pf(K, L) - vK - wL] \). This is maximum profit attainable given prices.

Properties of Profit Functions

1. Homogeneous of degree 1 – Inflation does not change quantities of inputs used and output produced, but profit will increase at the rate of inflation.
2. Nondecreasing in output price, \( p \) – If the firm does not change input use and output produced, profit will rise as \( p \) increases. If the firm changes input use or output in response to the increase in \( p \), it must be doing so to make even more profit. Therefore, if \( p \) increases, profit remains the same or increases; it cannot decrease for a profit-maximizing firm.
3. Nonincreasing in input prices – Similar to above discussion. When profit is maximized, a firm cannot reallocate input use without reducing profit. If \( v \) increases and the firm cannot reallocated resources to achieve higher profit or it would have allocated inputs differently before.

4. Convex in output prices – Average profits obtainable from two different output prices will be at least as high as profit obtained from the average of two output prices.

\[
\frac{\Pi(p_1, v, w) + \Pi(p_2, v, w)}{2} \geq \Pi\left[\frac{p_1 + p_2}{2}, v, w\right]
\]
The Envelope Theorem allows us to calculate the firm’s supply function and input demand functions by partially differentiating the Profit Function with respect to each of the prices as follows.

\[ \frac{\partial \Pi(p, v, w)}{\partial p} = q(p, v, w); \text{ the firm's supply function.} \]

\[ \frac{\partial \Pi(p, v, w)}{\partial v} = -K(p, v, w); \text{ the negative of the firm's derived demand function for capital (this is not contingent demand).} \]

\[ \frac{\partial \Pi(p, v, w)}{\partial w} = -L(p, v, w); \text{ the negative of the firm's derived demand function for labor (this is not contingent demand).} \]
Short-Run Producer Surplus

SMC = short-run supply curve (Set SMC=p and solve for q to get short-run supply function.)

The gain in short-run producer surplus from an increase in price from $p_1$ to $p_2$ is the area above the short-run supply curve between $p_1$ and $p_2$; the area $p_2ABp_1$.

With the price increase, producers gain $p_2-p_1$/unit of original production and they gain $p_2-p_? \ (p_? \ is \ between \ p_2 \ and \ p_1)$ on increased production between $q_1$ and $q_2$.

This change in producer surplus ends up being

\[
\text{Welfare gain} = \Pi(p_2, v, w) - \Pi(p_1, v, w).
\]

Short-run producer surplus at the prevailing market price is \(\Pi(p_1, v, w) - \Pi(p_0, v, w)\),

where $p_0$ is the shut-down price at minimum SAVC. Producer surplus is the extra return the producer makes from market transactions at the market price over and above what he/she would earn if nothing were produced. Finally, short-run producer surplus is:

\[
\Pi(p_1, v, w) - \Pi(p_0, v, w) = \Pi(p_1, v, w) - (-vK_1) = p_1 q_1 - vK_1 - wL_1 + vK_1 = p_1 q_1 - wL_1,
\]

because \(\Pi(p_1, v, w) = p_1 q_1 - vK_1 - wL_1\) and \(\Pi(p_0, v, w) = (-vK_1)\).

A firm’s short-run producer surplus is its total revenue minus its variable cost, which is what the firm gains at the market price ($p_1$) by producing rather than shutting down.
Profit Maximization and Input Use

Earlier we showed: \[ \pi(q) = p(q) \cdot q - C(q) \] but \[ q = f(K, L) \] and \[ C(q) = vK + wL \]

So, the profit-maximizing decision is a matter of choosing optimal amounts of the inputs K and L.

Max \[ \pi(K, L) = p \cdot f(K, L) - (vK + wL) \]

Two variables and no constraint.

(Assuming a price-taking firm in output and input markets. \( p \) is not a function of \( q \) and \( v \) and \( w \) are not functions of \( K \) and \( L \).)

FOC: \[ \frac{\partial \pi}{\partial K} = p \cdot \frac{\partial f}{\partial K} - v = 0, \quad \frac{\partial \pi}{\partial L} = p \cdot \frac{\partial f}{\partial L} - w = 0 \]

Define Marginal Revenue Product (MRP) as the marginal change in R for a small change in input use.

These FOCs mean that any input should be employed up to the point where its marginal contribution to revenue equals its marginal input cost (\( v \) or \( w \)).
Further, divide the second FOC by the first FOC to get:

\[
p \cdot \frac{\partial f}{\partial L} = \frac{w}{v} \quad \text{or} \quad \frac{\text{MP}_L}{\text{MP}_K} = \frac{w}{v} \quad \text{or} \quad \text{RTS}_{LK} = \frac{w}{v}
\]

The slope of isoquant equals the slope of isocost line.

We get the same solution as the two constrained optimization problems. This is the cost-minimizing combination of K and L for the given (optimal) output. It is also the output maximizing combination of K and L for the given (optimal) cost.

The SOC identify this optimal combination of K and L as giving maximum profit rather than a minimum or saddle point. The SOC are:

\[
\pi_{KK} < 0, \quad \pi_{LL} < 0, \quad \text{and} \quad \pi_{KK} \pi_{LL} - \pi_{KL}^2 > 0.
\]

If SOC are met, MC is increasing at q*.

Diminishing MP\(_K\) (f\(_{KK}\) < 0) and MP\(_L\) (f\(_{LL}\) < 0) mean that \(\pi_{KK}\) and \(\pi_{LL}\) < 0 (because \(\pi_{KK} = p f_{KK}\) and \(\pi_{LL} = p f_{LL}\)). But diminishing MP does not ensure increasing MC with two or more inputs.

The cross effects, \(\pi_{KL}\) and \(\pi_{LK}\), which are equal, must be small enough to be dominated by the own effects to ensure that MC is increasing.
The FOCs for profit maximization can be solved for the optimal combinations of K and L ($K^*$ and $L^*$) for any input and output prices ($p, v, w$).

Then $K^*$ and $L^*$ would be expressed as functions of $p$, $v$, and $w$ (for a given production function) to give “unconditional” derived input demand functions ($q$ is not constant).

$$K^* = K(p, v, w)$$

$$L^* = L(p, v, w)$$

Substitute $K^*$ and $L^*$ into the production function, $q = f(K, L)$, to get optimal output $q^*$.

$$q^* = q(K^*, L^*) = q(K(p, v, w), L(p, v, w)) = q(p, v, w)$$

This is the optimized production function, which is the firm’s supply function. This supply function shows how much $q$ will be supplied at different output and input prices. Is it a short-run supply function as shown in previous graphs?
Input Demand Functions

\[ K^* = K(p, v, w) \quad \partial L/\partial w \leq 0 \text{ always, and} \]
\[ L^* = L(p, v, w) \quad \partial K/\partial v \leq 0 \text{ always.} \]

**Single Input Case:**

Optimality (FOC) requires that \( w = p(MP_L) \). If \( p \) is fixed and \( w \) increases, \( MP_L \) must increase. Because \( MP_L \) is diminishing as \( L \) increases, less \( L \) must be used to cause an increase in \( p(MP_L) \) to maintain equality. Thus, if \( w \) increases, \( L \) must decline for FOC to continue to hold.

**Mathematically:**

Totally differentiate the FOC to get \( dw = p \cdot \frac{\partial f_L}{\partial L} \cdot \frac{\partial L}{\partial w} \cdot dw \)

or \( 1 = p \cdot f_{LL} \cdot \frac{\partial L}{\partial w} \) or \( \frac{\partial L}{\partial w} = \frac{1}{p \cdot f_{LL}} \leq 0 \)

The final inequality holds because \( f_{LL} \) is assumed to be \( \leq 0 \), i.e., \( MP_L = f_L \) diminishes, or remains unchanged, when \( L \) increases and vice versa.
Two Input Case

This situation is more complex than the single input case because the firm would need to adjust the amount of K as well as L in response to a change in w. The entire MP_K function moves when L changes. \( \Delta L \Rightarrow \Delta MP_K \).

When w changes, the effect on L can be decomposed into 1) the Substitution Effect and 2) the Output Effect.

**Substitution Effect**

If q is held constant while w decreases, there will be a substitution of L for K in the optimum input mix. Because the minimum cost use of K and L requires that RTS_{LK} = w/v, a decrease in w will cause a new optimal point at a lower RTS (less K and more L). The substitution effect will be negative.

An increase in w will decrease L (and increase K) and a decrease in w will increase L (and decrease K).

The change in L in going from A to B is the substitution effect = \( L_2 - L_1 \).
Output Effect for a Normal Input

The Output Effect is **negative**. That is, a reduction in \( w \) will reduce MC of output, which will cause an increase in the profit-maximizing \( q \).

For a normal input, a decrease (increase) in \( w \) reduces (increases) MC, causing \( q^* \) to increase (decrease).

Looking at the isoquants above, the firm moves to \( q_2^* \) and increases L use from \( L_2 \) to \( L_3 \), so a decrease in \( w \) causes an increase in L use. In the above case, the output effect is negative, \( \partial L/\partial w < 0 \).

If the MC curves for all firms in the industry decrease with a decrease in \( w \), the industry supply curve \( (S=\Sigma MC) \) would shift outward. This would cause \( P \) to fall and industry output would also be higher as shown in the graph.

In the firm and industry case, \( \partial L/\partial w < 0 \). Both substitution and output effects are negative.

**Summary:** Change in L from a decrease in \( w \) in going from A to C = total effect = \( L_3 - L_1 \) equals the change in L in going from A to B = substitution effect = \( L_2 - L_1 \) plus the change in L in going from B to C = output effect = \( L_3 - L_2 \).
Output Effect for an Inferior Input

The Output Effect is also **negative**. That is, a reduction in w will increase MC of output, which will cause an increase in the profit-maximizing q.

For an inferior input, a decrease (increase) in w increases (reduces) MC, causing q* to decrease (increase).

Looking at the isoquants above, the firm moves from q_1* to q_0* and increases L use from L_2 to L_3, so a decrease in w causes an increase in L use. In the above case, the output effect is negative, \( \partial L / \partial w < 0 \).

**Summary:** The change in L from a decrease in w in going from A to C = total effect = \( L_3 - L_1 \) equals the change in L in going from A to B = substitution effect = \( L_2 - L_1 \) plus the change in L in going from B to C = output effect = \( L_3 - L_2 \). For an inferior input, a decrease in w causes L to increase because of the substitution effect and to increase further because of the output effect; thus, the total effect is negative.
Cross-Price Effects (Effect of $\Delta w$ on $K$?)

In the graph, the cross effect of a decrease in $w$ is:

1) The cross-substitution effect caused a decline in $K$:

Point A to point B. $K_1$ to $K_3 \Rightarrow \frac{\partial K}{\partial w} \bigg|_{q=q^*} > 0$

2) The output effect caused an increase in $K$ due to $\Delta q$:

Point B to point C. $K_3$ to $K_2 \Rightarrow \frac{\partial K}{\partial w} < 0$

Thus, the total effect on $K$ of a decrease in $w$ could be either positive or negative depending on shape of isoquants and the amount $q$ increases when $w$ decreases ($\frac{\partial K}{\partial w} \leftrightarrow 0$).
Mathematical Derivation of Substitution and Output Effects.

Begin with the FOC’s for choosing K and L to maximize $\pi$: $v = p(MP_K)$ and $w = p(MP_L)$. Solving these simultaneously shows that the profit-maximizing amounts of K and L are functions of $p$, $v$, and $w$ for a given production function. Thus, $K^* = K(p,w,v)$ and $L^* = L(p,w,v)$. These are derived demand functions for the inputs. $p$ and either $w$ or $v$ are shifters of the $K^* = f(v|p,w)$ or $L^* = f(w|p,v)$ demand curves. We will use L as an example and show how changes in $w$ affect $L^*$. Remember that, at the profit-maximizing choice of L, derived demand for L equals contingent demand for L:

$L(p,v,w) = L^c(v,w,q)$.

Differentiate both sides to get:

$$\frac{\partial L(p, v, w)}{\partial w} = \frac{\partial L^c(v, w, q)}{\partial w} + \frac{\partial L^c(v, w, q)}{\partial q} \cdot \frac{\partial q}{\partial w}.$$  

This says that the total effect of a change in $w$ on demand for L has two parts: 1) the change in contingent labor demand holding $q$ constant (substitution effect), and 2) the change in contingent labor demand from a change in the level of output (output effect).

The first term on the right-hand side is negative because of strictly convex isoquants.
The second term on the right-hand side (the output effect) is:
\[ \frac{\partial L^c}{\partial q} \cdot \frac{\partial q}{\partial w} = \frac{\partial L^c}{\partial q} \cdot \frac{\partial q}{\partial MC} \cdot \frac{\partial MC}{\partial w} ; \text{ where } q = q(MC) \text{ at } P = MC. \]

\[ \frac{\partial q}{\partial MC} < 0 \]

For a normal input, \( \frac{\partial L^c}{\partial q} > 0 \) and \( \frac{\partial MC}{\partial w} > 0, \)

For an inferior input, \( \frac{\partial L^c}{\partial q} < 0 \) and \( \frac{\partial MC}{\partial w} < 0, \)

so the output effect is always negative.

In any case, the output effect is negative. This result along with the negative substitution effect combine to give a negative total effect on input demand resulting from an input price change.

\[ \frac{\partial L(p, v, w)}{\partial w} \leq 0 \text{ always.} \]

The Giffen paradox cannot occur in the demand for inputs.