Chapter 4

Utility Maximization and Choice
• Maximization of utility subject to a budget constraint – We will show how an individual with a given preference system, faced with fixed prices and a fixed budget (income), chooses among goods.

• **Budget Constraint**
  
  I is income in dollars/period (fixed for now). x and y are quantities of two goods/period. p_x and p_y are prices/unit of goods x and y (fixed for now).

  For utility maximization, I=expenditures, i.e.,
  
  \[ I = p_x x + p_y y. \]
For U max: \( I = p_x x + p_y y \)

Expenditures

Utility maximization implies \( I=E \) because all income will be spent.

Budget constraint shows all combinations of \( x \) and \( y \) that can be purchased with income of \( I \).

Solve the budget constraint for \( y \).

\[
y = \frac{I}{p_y} - \frac{p_x}{p_y} x
\]

Slope = \(-\frac{p_x}{p_y}\)

The intercepts show the quantities of \( x \) and \( y \) when all income is spent on a single good.

\[
\frac{I}{p_x}, \text{ e.g., } I = \$50, \ p_x = \$5/\text{unit} \Rightarrow \frac{I}{p_x} = 10 \text{ units of } x.
\]
Combine the budget constraint, which contains income and price information (market info.), with the indifference curves, which contain consumer preference information (psychological info.).

Point C is the optimum point because it is where the budget constraint allows achieving the highest indifference curve. Cannot reach U₃. Can improve on U₁ by changing the combination of x and y.

Maximum utility is U₂ (U*). Point C is the tangency between the indifference curve and the budget constraint. Therefore, at C where utility is maximized, the slopes of the indifference curve and the budget constraint are equal.
At Point C, \(-\frac{p_x}{p_y} = \frac{dy}{dx}\) \(\left| U^* = -\text{MRS}_{xy}\right\)

The FOCs for Utility Maximization are 1) spend all income (I=E) and 2) choose the combination of goods where \(\text{MRS}_{xy}\) equals the price ratio of the goods.

\[
\text{MRS}_{xy} = -\frac{dy}{dx} \text{(preferences)} = -\frac{dy}{dx} \text{(income)} = \frac{p_x}{p_y}
\]

The SOC (sufficient condition) is that \(\text{MRS}_{xy}\) is diminishing throughout the indifference curve (strictly convex indifference curve, strictly quasi-concave utility function).
The FOC stated previously applies only to “interior” solutions, not to “corner” solutions.

The optimal indifference curve is not tangent to the budget constraint.

At Point C, $\text{MRS}_{xy} > \frac{p_x}{p_y}$, so consume $x = \frac{I}{p_x}$ and no $y$.

If $\text{MRS}_{xy} < \frac{p_x}{p_y}$, consume $y = \frac{I}{p_y}$ and no $x$. 
In Mathematical Format

**Max**: $U = U(x_1, x_2, \ldots, x_n)$

**Utility function**

**st**: $I = p_{x_1}x_1 + p_{x_2}x_2 + \ldots + p_{x_n}x_n$

Constant on LHS in explicate form.

Implicit form.

$$\mathcal{L} = U(x_1, x_2, \ldots, x_n) + \lambda (I - p_{x_1}x_1 - p_{x_2}x_2 - \ldots - p_{x_n}x_n)$$

These are the necessary conditions. Solve the FOC simultaneously for $x_1^* , \ldots , x_n^*$.

If MRS is diminishing (SOC — $U$ is strictly quasi-concave; indifference curves are strictly convex), the sufficient condition is satisfied.

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda p_{x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda p_{x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial U}{\partial x_n} - \lambda p_{x_n} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_{x_1}x_1 - p_{x_2}x_2 - \ldots - p_{x_n}x_n = 0$$

FOC
Manipulate the FOC

\[ \frac{\partial U}{\partial x_i} - \lambda p_{x_i} = 0 \quad \text{all } i \Rightarrow \frac{\partial U}{\partial x_i} = \lambda \quad \text{all } i. \]

Thus,
\[ \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} \quad \text{for } i \neq j \]
\[ \frac{\partial U}{\partial x_i} = \frac{p_{x_i}}{p_{x_j}} \quad \text{for } i \neq j. \]

Thus, \[ \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} \quad \text{for } i \neq j \]
\[ \frac{p_{x_i}}{p_{x_j}} \]

and
\[ \frac{\partial U}{\partial x_i} = \frac{p_{x_i}}{p_{x_j}} \quad \text{for } i \neq j. \]

\[ \text{MU}_{x_i} = \text{MU}_{x_j} \]
\[ \frac{p_{x_i}}{p_{x_j}} \]

MU/$ spent on i = MU/$ spent on j.

If \( \text{MU}_{x_1} = 2\text{MU}_{x_2} \), then \( p_{x_1} = 2p_{x_2} \).

\[ \frac{\text{MU}_{x_i}}{\text{MU}_{x_j}} = \frac{p_{x_i}}{p_{x_j}} \quad \text{tangency point. Because} \quad \frac{\text{MU}_{x_i}}{\text{MU}_{x_j}} = \text{MRS}_{x_i,x_j}, \]

\[ \frac{\text{MU}_{x_i}}{\text{MU}_{x_j}} = \text{MRS}_{x_i,x_j} = -\frac{dx_j}{dx_i} = \frac{p_{x_i}}{p_{x_j}} \quad \text{Psychic rate of trade-off equals the market rate of trade-off between goods at the optimum.} \]
Also...

\[ \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} = \lambda \]

from above.

\[ \frac{\text{MU} x_i}{p_{x_i}} = \frac{\text{MU} x_j}{p_{x_j}} = \lambda \]

If \( \text{MU}_{x_i} = 2 \), \( p_{x_i} = $1 \), and \( p_{x_j} = $0.5 \), then \( \text{MU}_{x_j} = 1 \). Thus, $1 spent on either good yields the same utility, which equals \( \lambda \)! Thus, \( \lambda \) equals the MU of money (I).

The ratio of \( \text{MU}_{x_i} \) to \( p_{x_i} \) is the same for all goods. One dollar gives the same addition to utility no matter where it is spent (if not, the consumer would not be at optimum).
\( \lambda \) is the marginal benefit from relaxing the income constraint by $1 (increasing I by $1)! Shows MU obtained from last dollar spent on all goods. From the FOC,

\[
p_{x_i} = \frac{MU_{x_i}}{\lambda} = \frac{MU_{x_i}}{MU_I}
\]

This equation says that \( p_{x_i} \) equals the ratio of \( MU_{x_i} \) divided by the \( MU_I \) at optimum utility. \( MU_{x_i}/MU_I \) must equal the rate at which the consumer can turn money into goods or vice versa, which equals \( p_{x_i} \) (for each \( i \)).
Corner Solutions

When \( p_{x_i} > \frac{MU_{x_i}}{MU_I} = \frac{MU_{x_i}}{\lambda} \) for all units of \( x_i \), no \( x_i \) will be purchased!

\[ p_{x_2} > \frac{MU_{x_2}}{MU_I} \]

\[ -\frac{dx_2}{dx_1} = MRS_{x_1x_2} = \frac{MU_{x_1}}{MU_{x_2}} > \frac{p_{x_1}}{p_{x_2}} \]

No \( x_2 \) is purchased because the price of \( x_2 \) is too high compared with what a unit of \( x_2 \) is worth to the consumer.
Indirect Utility Function – It is the utility function expressed at optimal levels of $X_i$, at $X_i^*$. Simultaneous solution of the FOC for a constrained utility maximization problem will yield the optimal quantities of $X_i$ expressed in terms of $p_{x_i}$ and I as a demand function.
Deriving Indirect Utility Function

Max: \( U = xy \)

st: \( I - p_x x - p_y y = 0 \)

\[ \ell = xy + \lambda (I - p_x x - p_y y) \]

\[ \frac{\partial \ell}{\partial x} = y - \lambda p_x = 0 \]
\[ \frac{\partial \ell}{\partial y} = x - \lambda p_y = 0 \]
\[ \frac{\partial \ell}{\partial \lambda} = I - p_x x - p_y y = 0 \]

Divide second by first

\[ \frac{x}{y} = \frac{\lambda p_y}{\lambda p_x} \text{ or } \frac{x}{y} = \frac{p_y}{p_x} \]

This direct utility function is Cobb-Douglas with \( \alpha = \beta = 1 \).

Expenditures on \( x = \) expenditures on \( y \) when \( \alpha = \beta \).

or \( p_x x = p_y y \).
Substitute into the constraint:

\[ I = p_x x + p_x x = 2p_x x \quad \text{and} \quad I = p_y y + p_y y = 2p_y y \]

Solve for \( x \) and \( y \) to get the demand functions.

\[ x^* = \frac{I}{2p_x} \quad \text{and} \quad y^* = \frac{I}{2p_y} \]

Changes in \( I \) or prices change the demand functions \((x^*, y^*)\).

Substitute \( x^* \) and \( y^* \) into \( U = U(x, y) \), or \( U^* = x^* y^* \).

\[ U^* = V\left(\frac{I}{2p_x}, \frac{I}{2p_y}\right) \quad \text{or} \quad U^* = V(p_x, p_y, I) \]

Indirect Utility Function!

Optimum \( U \) depends indirectly on prices and \( I \)!

\[ U^* = x^* y^* = U^* = \frac{I}{2p_x}\left(\frac{I}{2p_y}\right) = \frac{I^2}{4p_x p_y} \]

Indirect utility function for this case.
• The dual to the constrained utility maximization problem gives the expenditure function, which is expenditures expressed in terms of the optimal levels of \( x^* \) and \( y^* \).

• Earlier discussion of the Primal (max \( U \) st I constraint to get \( x^* \), \( y^* \), and \( U^* \)) suggests that the dual would be:

\[
E = I \text{ if all income is allocated.}
\]

**Min:** \( E = p_{x_1} x_1 + p_{x_2} x_2 + \ldots + p_{x_n} x_n \)

st: \( U^* = U(x_1, x_2, \ldots, x_n) \)

• Solve the FOCs simultaneously to get \( x^* \), \( y^* \). Substitute this optimal solution into the objective function above to get \( E^* \), which is the expenditure function.

\[
E^* = p_{x_1} x_1^* + p_{x_2} x_2^* + \ldots + p_{x_n} x_n^*
\]
Minimization Problem

Min $E = p_x x + p_y y$

$s.t. U^* = f(x, y)$

$U^*$ is a constant.

Many different combinations of $x$ and $y$ give $U^*$, but minimum expenditures occurs when $MRS_{xy} = \frac{p_x}{p_y}$. 
• The optimal point \((x^*, y^*, \text{ and } E^*)\) will be identical (in terms of \(x^*\) and \(y^*\)) to the primal solution. \(x^*\) and \(y^*\) will depend on \(p_x, p_y\) and on the \(U^*\) level required. Prices perform the same function as earlier, but utility \((U)\) has now replaced income \((I)\) as the constraint.

• Earlier (from primal) we discussed how optimal utility was dependent on prices and income: Maximum utility = \(U^* = V(p_x, p_y, I)\). This is the Indirect Utility Function!
Now minimal expenditures (from the Dual) can be expressed as a function of prices and utility:

\[ E^* = \text{Min } E = E(p_x, p_y, U) \]

The expenditure function is the inverse of the indirect utility function if \( E = I \). For example:

\[ U^* = x^* y^* = \frac{I^2}{4p_x p_y} = \frac{E^2}{4p_x p_y} \quad \text{if } I = E. \]

Solve for \( E \):

\[ E^2 = 4U^* p_x p_y \Rightarrow E^* = 2U^{.5} p_x^{.5} p_y^{.5}. \]

\( E^* \) is called the **Expenditure Function** (will not always use the *). Minimum expenditures is expressed as a function of prices and the utility level.
Properties of Expenditure Functions

• Homogeneity of degree 1 in prices – Total expenditures increase by \( t \) if all prices are multiplied by \( t \), i.e., a doubling of all prices doubles expenditures.

• Expenditure functions are nondecreasing in prices – If the price of any one good increases, expenditures cannot decrease and would likely increase because of expenditure minimization, mathematically \( \frac{\partial E}{\partial P_i} \geq 0 \) for every good \( i \).

• Expenditure functions are concave in prices -

\( p_1^* \) is the initial price for which expenditures are \( E(p_1^*\ldots) \). For prices higher or lower than \( p_1^* \), if the person continued to purchase the same bundle of goods, expenditures would be on \( E^{\text{pseudo}} \). But because of expenditure minimization, the person would change the bundle of goods consumed and expenditures would follow \( E(p_1\ldots) \).