Chapter 18

Expected Utility and Risk Aversion

Up to now we have assumed no uncertainty; any course of action had a certain utility. This is not always realistic. We will now deal with a case where the purchase of a good does not guarantee a certain outcome.
Expected Value

The expected value of any game is the sum of all possible outcomes weighted by their respective probabilities.

\[ E(X) = \pi_1 x_1 + \pi_2 x_2 + \ldots + \pi_n x_n = \sum_{i=1}^{n} \pi_i x_i \quad (\sum_{i=1}^{n} \pi_i = 1) \]

- An actuarially fair game is one with an expected value equal to zero or one that costs its expected value for the right to play. In other words, a fair game is one in which:
  - The weighted sum of the outcomes equals zero, or
  - To play, a person must pay an amount equal to the positive expected value or receive an amount equal to the negative expected value of the game.

\[ \sum_{i=1}^{n} \pi_i x_i = 0 \quad \text{or} \quad \sum_{i=1}^{n} \pi_i x_i - \text{sum} = 0 \quad \text{or} \quad \sum_{i=1}^{n} \pi_i x_i + \text{sum} = 0 \]

Pay \quad \text{Receive}
St. Petersburg Paradox

People are generally unwilling to play fair games, especially games with very large expected values. Individuals are not usually willing to pay large amounts to play. They are not even willing to pay much smaller amounts than the expected value to play if the expected value is very large.

\[
\begin{align*}
0.5 \text{ probability of winning } & \$3000 \\
0.5 \text{ probability of loosing } & \$1000
\end{align*}
\]

\[E(X) = 0.5(3000) + 0.5(-1000) = 1000.\]

If a person paid $1000, it would be a fair game, \(0.5(3000) + 0.5(-1000) - 1000 = 0\). Why would most people not be willing to pay $1000 for the chance to play the game?

Bernoulli’s solution was that the Marginal Utility of money declines. Thus, the individual would not be willing to pay the game’s expected value because each additional dollar brings smaller and smaller increases in utility as income increases; each dollar that could be won would be worth less than each dollar that could be lost or paid to play the game.
Expected Utility Maximization
(rather than expected value maximization)

\[ E(U(X)) = \pi_1 U(x_1) + \pi_2 U(x_2) + \ldots + \pi_n U(x_n) = \sum_{i=1}^{n} \pi_i U(x_i) \]

- Individuals will make choices that maximize the expected utility of the various choices. They will choose the alternative that maximizes expected utility rather than expected value.

Risk
- “Risk” refers to variability of outcomes of a given activity. More variability implies more risk. For example, more people would be willing to flip a coin for a dollar than would be willing to flip a coin for $1000. The expected value is 0 for both games.
- People are usually risk averse because of diminishing MU of money. They expect less utility from the dollar they might win than the utility from the dollar they might lose. They will choose the activity with the lowest risk when all activities have the same expected value.
**Example**

*Suppose* you were offered a 50-50 chance of winning or loosing $h$ wealth (fair) or a 50-50 chance of winning or loosing $2h$ wealth (also fair). Assume MU of wealth is declining; $U'(W) > 0$ and $U''(W) < 0$.

Draw $U(W)$ and begin at certain wealth equal $W^*$ giving $U(W^*)$ (if don’t play the game).

$EV_1 = 0.5(W^* - h) + 0.5(W^* + h)$ is the expected value of the first game and expected utility is $EU^h = 0.5U(W^* - h) + 0.5U(W^* + h)$.

$EV_2 = 0.5(W^* - 2h) + 0.5(W^* + 2h)$ is the expected value of the second game and expected utility is $EU^{2h} = 0.5U(W^* - 2h) + 0.5U(W^* + 2h)$.

$EV_1 = EV_2 = W^*$ because of 50-50 chance of winning or losing equal amounts. What would this graph look like if 25-75 chance?
Conclusion:

• An individual will choose the first bet over the second because the first gives higher expected utility (smaller variance of outcomes), but neither bet is preferred to certain wealth of \( W^* \) from not playing the game, because of declining marginal utility of wealth (concavity of \( U(W) \)).

• If given a choice between playing the game or paying a sum not to play, an individual would be willing to pay money to avoid playing as long as utility of certain wealth minus the payment from not playing the game is greater than expected utility from playing the game. This is why we buy insurance against a fair game. He/she would pay up to \( P_1 \) not to play the first game and up to \( P_2 \) not to play the second.

• The individual will give up money until certain utility of wealth, \( U(W^* - P_1) \), equals expected utility from playing the game, \( EU^h \).
Example:

- Assume a natural log utility of wealth function.
- Current wealth is $W = 150,000$.
- A 0.20 chance of losing a $30,000 car.
- Then the expected value of wealth is
  \[ EV = 0.20(120,000) + 0.80(150,000) = 144,000. \]
- Expected loss is
  \[ EL = 0.20(30,000) + 0.80(0) = 6,000; \]
  \[ EL = W - EV = 150,000 - 144,000 = 6,000. \]
$U_1$ = utility if no loss and don’t pay for insurance.

$U_2$ = utility if loss and don’t pay for insurance.

$U_3$ = utility of certain $144,000$ when pay expected loss (= $6,000) for insurance = \(\ln(144,000) = 11.87757\).

$EU_4$ = expected utility of uncertain $144,000$ when don’t pay $6,000$ for insurance = \(0.20\ln(120,000) + 0.80\ln(150,000) = 11.87376\).

$U_3 > EU_4$ so better off buying insurance to cover expected loss.

**Find** $P$ (maximum amount the person is willing to pay for insurance) by setting certain utility (if pay insurance, $P$) equal to expected utility if do not pay ($EU_4$). \(\ln(150,000-P) = 11.87376\). Solve this for $P$: \(150,000-P = e^{11.87376}\) to get $P = $6,547.

$P = $6,000 to pay for expected loss + $547 for administrative costs and profit for the insurance company.

The person is just as well off paying $6,547 to have certain wealth of $143,453 as not paying and having uncertain expected wealth of $144,000.
Risk Aversion

- An individual who is risk averse always refuses fair bets or is willing to pay to avoid a fair bet.
- People will be risk averse if MU of money or wealth is declining!
- People are risk lovers if MU of money or wealth is increasing as in the graph below.

This person would prefer to play the game for an expected wealth of $EV_1$ and expected utility of $EU_2$ rather than have certain wealth of $W_1$ and certain utility of $U_1$. This person would have to be paid at least $P$ for not gambling to give the same utility ($U_2$) as gambling ($EU_2$) or the person would be willing to play an unfair game with expected value as low as $EV_1 - P^*$, where $P^*$ is the expected loss from playing the game.
Pratt’s Measure of absolute risk aversion

\[
r(W) = - \frac{U''(W)}{U'(W)} = - \frac{\partial^2 U}{\partial W^2} \quad \text{(curvature)}
\]

\[
\quad \frac{\partial U}{\partial W} \quad \text{(slope)}
\]

\[
\begin{align*}
\text{larger } r(W) & \text{ means more risk averse!} \\
\text{Denominator } & > 0. \\
\text{Numerator } & < 0. \\
\text{The negative sign makes } r(W) & \text{ positive.}
\end{align*}
\]

- Denominator \(> 0\).
- Numerator \(< 0\).
- The negative sign makes \(r(W)\) positive.

\[P = kr(W), \text{ ie., the more risk averse, the more a person is willing to pay to avoid the fair game.}\]

\(r(W)\) is proportional to an individual’s willingness to pay for insurance. The text shows that willingness to pay to avoid a fair bet is proportional to \(r(W)\). That is, \(P = kr(W)\); but \(r(W)\) may not be constant in relation to wealth.
Risk Aversion and Wealth

Quadratic Utility:
\[ U(W) = a + bW + cW^2, \text{ where } b > 0, c < 0. \]
\[ r(W) = -\frac{U''(W)}{U'(W)} = \frac{-2c}{b + 2cW}; \text{ risk aversion increases with wealth.} \]

Logarithmic Utility:
\[ U(W) = \ln(W), \text{ where } W > 0. \]
\[ r(W) = -\frac{U''(W)}{U'(W)} = \frac{1}{W}; \text{ risk aversion decreases with wealth.} \]

Exponential Utility:
\[ U(W) = -e^{-AW} = -\exp(-AW), \text{ where } A > 0. \]
\[ r(W) = -\frac{U''(W)}{U'(W)} = \frac{A^2 e^{-AW}}{Ae^{-AW}} = A; \text{ risk aversion is constant with wealth,} \]
ie.e., Exponential Utility exhibits constant absolute risk aversion.
Relative Risk Aversion

\[ rr(W) = Wr(W) = -W \frac{U''(W)}{U'(W)} \]

constant, then the individual has constant relative risk aversion.

- If \( rr(W) \) is constant, a person’s willingness to pay for insurance against a fair bet is inversely proportional to wealth.
- If \( rr(W) \) is constant, as \( W \) increases, \( r(W) \) must decrease proportionally for \( rr(W) \) to remain constant, so the individual’s willingness to pay to avoid a fair bet, \( P=kr(W) \), declines proportionally as \( W \) increases.
- Wealthy people are willing to pay less for insurance against a fair bet.
- Poor are willing to pay more for insurance.