Unit 13: Analysis of Multifactor Experiments

Statistics 571: Statistical Methods
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Analysis of Multifactor Experiments

Table 13.1 Data from a Balanced Two-Way Layout

Factor A Levels	Factor B Levels 1	2	$\cdots$	j	$\cdots$	b
1	$y_{111}, \ldots, y_{11n}$	$y_{121}, \ldots, y_{12n}$	$\cdots$	$y_{1j1}, \ldots, y_{1jn}$	$\cdots$	$y_{1b1}, \ldots, y_{1bn}$
2	$y_{211}, \ldots, y_{21n}$	$y_{221}, \ldots, y_{22n}$	$\cdots$	$y_{2j1}, \ldots, y_{2jn}$	$\cdots$	$y_{2b1}, \ldots, y_{2bn}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$y_{i11}, \ldots, y_{i1n}$	$y_{i21}, \ldots, y_{i2n}$	$\cdots$	$y_{ij1}, \ldots, y_{ijn}$	$\cdots$	$y_{ib1}, \ldots, y_{ibn}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a	$y_{a11}, \ldots, y_{a1n}$	$y_{a21}, \ldots, y_{a2n}$	$\cdots$	$y_{aj1}, \ldots, y_{ajn}$	$\cdots$	$y_{ab1}, \ldots, y_{abn}$

Balanced two-way layout:

- n = replicates
- Two factors: So there are *ab* treatment combinations
- Completely randomized design

EXAMPLE 13.1 (BONDING STRENGTH OF CAPACITORS: PARAMETER ESTIMATES)

Capacitors are bonded to a circuit board used in high voltage electronic equipment. Engineers designed and carried out an experiment to study how the mechanical bonding strength of capacitors depends on the the type of substrate (factor A) and the bonding material (factor B). There were three types of substrates: aluminium oxide (Al_2O_3) with bracket, Al_2O_3 without bracket, and beryllium oxide (BeO) without bracket. Four types of bonding material were used: Epoxy I, Epoxy II, Solder I, and Solder II. Four capacitors were tested at each factor level combination. Simulated bonding strength data are given in Table 13.2. Calculate the estimates of the parameters of model (13.3) for these data.

Table 13.2 Bonding Strength of Capacitors (Balanced Data)

Substrate	Bonding Material			
	Epoxy I	Epoxy II	Solder I	Solder II
Al_2O_3	1.51, 1.96	2.62, 2.82	2.96, 2.82	3.67, 3.40
w/o bracket	1.83, 1.98	2.69, 2.93	3.11, 3.11	3.25, 2.90
Al_2O_3	1.63, 1.80	3.12, 2.94	2.91, 2.93	3.48, 3.51
w/ bracket	1.92, 1.71	3.23, 2.99	3.01, 2.93	3.24, 3.45
BeO	3.04, 3.16	1.91, 2.11	3.04, 2.91	3.47, 3.42
	3.09, 3.50	1.78, 2.25	2.48, 2.83	3.31, 3.76

This is a completely randomized design. The 48 observations were done in random order

Model for Balanced Two-Way Layout

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

$$(i = 1,...a;\ j = 1,...,b;\ k = 1,...,n)$$

$$\sum_{i=1}^{a} \tau_i = 0,\ \sum_{j=1}^{b} \beta_j = 0,$$

$$\sum_{i=1}^{a} (\tau\beta)_{ij} = 0 \text{ for all } j$$

$$\sum_{j=1}^{b} (\tau\beta)_{ij} = 0 \text{ for all } i$$

$(\tau\beta)_{ij}$ is called the (i, j)th row-column interaction

The ε_{ijk}'s are independent and identically distributed (i.i.d.) $N(0,\sigma^2)$ random errors.

Interactions

$$\mu_{ij} - \mu_{i'j} = \left(\mu + \tau_i + \beta_j + (\tau\beta)_{ij}\right) - \left(\mu + \tau_{i'} + \beta_j + (\tau\beta)_{i'j}\right)$$

$$= \left(\tau_i - \tau_{i'}\right) + \left((\tau\beta)_{ij} - (\tau\beta)_{i'j}\right) \text{ which depends on } j$$

Similarly,

$$\mu_{ij} - \mu_{ij'} = \left(\beta_j - \beta_{j'}\right) + \left((\tau\beta)_{ij} - (\tau\beta)_{ij'}\right) \text{ which depends on } i$$

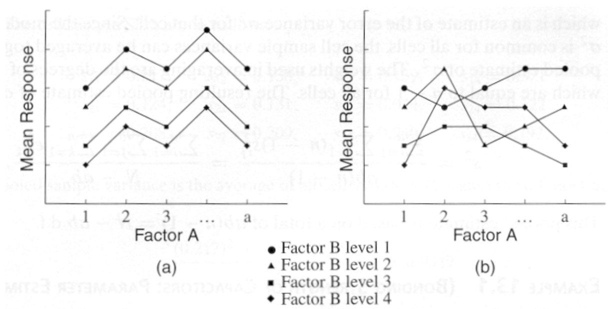

Figure 13.1 Interaction Plots: (a) No AB interaction; (b) AB interaction present

With no interaction one can understand the variation in y in terms of the A and B effects. With an interaction knowing the A and B effect is not enough.

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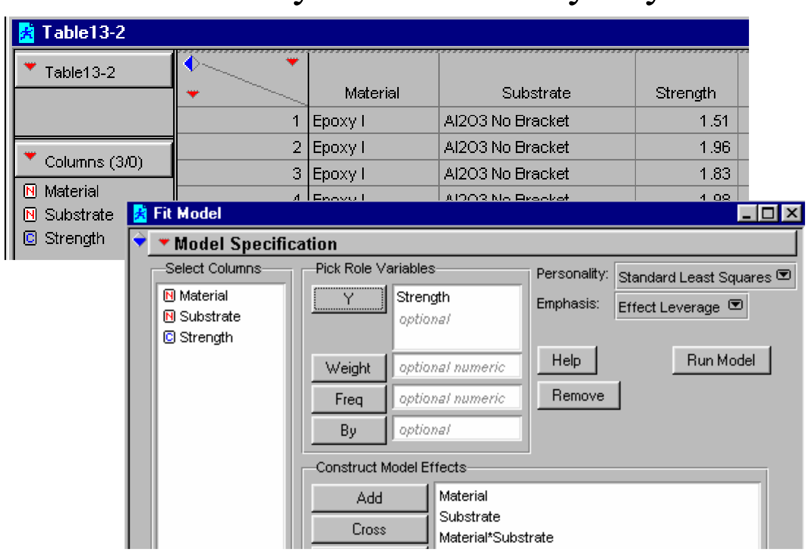

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Summary of Fit

RSquare	0.928168
RSquare Adj	0.906219
Root Mean Square Error	0.186864
Mean of Response	2.800417
Observations (or Sum Wgts)	48

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	11	16.242742	1.47661	42.2879
Error	36	1.257050	0.03492	Prob > F
C. Total	47	17.499792		<.0001

The model (i.e. substrate, material, and their interaction) explains 93% of the variation in bonding strength

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SSModel = SSMaterial + SSSubstrate +SSMaterial*Substrate

Effect Tests

Source	Nparm	DF	Sum of Squares	F Ratio	Prob > F
Material	3	3	8.4605083	80.7654	<.0001
Substrate	2	2	0.1953167	2.7968	0.0743
Material*Substrate	6	6	7.5869167	36.2130	<.0001

Sequential (Type 1) Tests

Source	Nparm	DF	Seq SS	F Ratio	Prob > F
Material	3	3	8.4605083	80.7654	<.0001
Substrate	2	2	0.1953167	2.7968	0.0743
Material*Substrate	6	6	7.5869167	36.2130	<.0001

Notice that the interaction is significant

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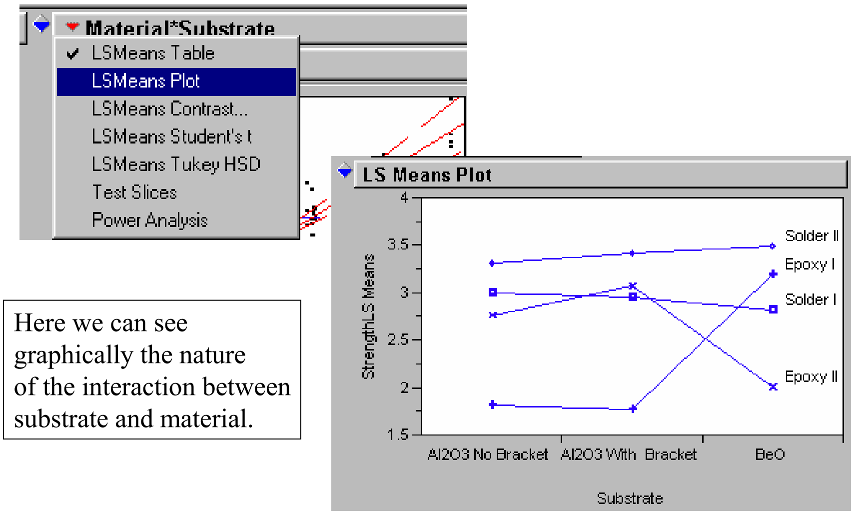
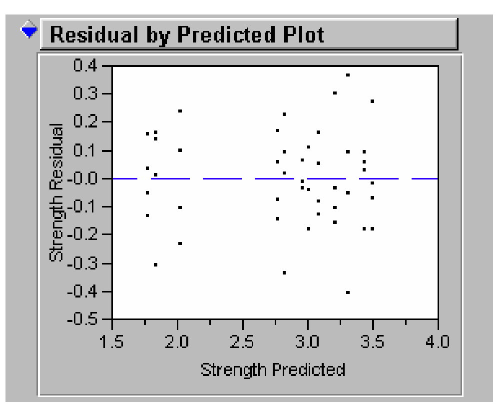


Here we can see graphically the nature of the interaction between substrate and material.

Residual Diagnostic Plots



The usual residual diagnostic plots can (should) be used with the two-way layout

Do You Need to Know More?

573 Design of Experiments (3) One-way ANOVA, multiple range tests, equal and unequal variances, transformations; factorial experiments, completely randomized designs, analysis of covariance, split-plot and nested designs, fractional factorials, sequential designs. Prereq: 571. Sp

"Design and Analysis of Experiments, Fifth Edition" by Douglas C. Montgomery. Wiley.