Bayesian Modeling of Accelerated Life Tests with Random Effects

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Abstract

We show how to use Bayesian modeling to make inference on the basis of data from an accelerated life test where the test units come from different groups (such as batches) and the group effect is random and significant both statistically and practically. Our approach can handle multiple random effects and several accelerating factors. However, we present our approach on the basis on an important application concerning pressure vessels wrapped in Kevlar 49 fibers where the fibers of each vessel comes from a single spool and the spool effect is random. We show how Bayesian modeling using Markov chain Monte Carlo (MCMC) methods can be used to easily answer questions of interest in accelerated life tests with random effects that are not easily answered with more traditional methods. For example, we can predict the lifetime of a pressure vessel wound with a Kevlar 49 fiber either from a spool used in the accelerated life test or from another random spool from the population of spools. We comment on the implications that this analysis has on the estimates of reliability (and safety) for the Space Shuttle, which has a system of 22 such pressure vessels. Our approach is implemented in the freely available WinBUGS software so that readers can apply the method to their own data.

Failure Time in hours of Kevlar 49 Wrapped Pressure Vessels

Stress	Spool	F-Time	Stress	Spool	F-Time	Stress	Spool	F-Time	Stress	Spool	F-Time
29.7	2	2.2	29.7	5	243.9	27.6	2	694.1	25.5	1	11487.3
29.7	7	4.0	29.7	4	254.1	27.6	4	876.7	25.5	5	11727.1
29.7	7	4.0	29.7	1	444.4	27.6	1	930.4	25.5	4	13501.3
29.7	7	4.6	29.7	8	590.4	27.6	6	1254.9	25.5	1	14032.0
29.7	7	6.1	29.7	8	638.2	27.6	4	1275.6	25.5	4	29808.0
29.7	6	6.7	29.7	1	755.2	27.6	4	1536.8	25.5	1	31008.0
29.7	7	7.9	29.7	1	952.2	27.6	1	1755.5	23.4	7	4000.0
29.7	5	8.3	29.7	1	1108.2	27.6	8	2046.2	23.4	7	5376.0
29.7	2	8.5	29.7	4	1148.5	27.6	4	6177.5	23.4	6	7320.0
29.7	2	9.1	29.7	4	1569.3	25.5	6	225.2	23.4	3	8616.0

The stress applied to the Kevlar 49 strands in the pressure vessels are in MPa or MegaPascals.

Failure Time in hours of Kevlar 49 Wrapped Pressure Vessels

29.7	2	10.2	29.7	4	1750.6	25.5	7	503.6	23.4	5	9120.0
29.7	3	12.5	29.7	4	1802.1	25.5	3	1087.7	23.4	2	14400.0
29.7	5	13.3	27.6	3	19.1	25.5	2	1134.3	23.4	6	16104.0
29.7	7	14.0	27.6	3	24.3	25.5	2	1824.3	23.4	5	20231.0
29.7	3	14.6	27.6	3	69.8	25.5	2	1920.1	23.4	6	20233.0
29.7	6	15.0	27.6	2	71.2	25.5	2	2383.0	23.4	5	35880.0
29.7	3	18.7	27.6	3	136.0	25.5	3	2442.5	23.4	1	41000.0*
29.7	2	22.1	27.6	2	199.1	25.5	8	2974.6	23.4	1	41000.0*
29.7	7	45.9	27.6	2	403.7	25.5	2	3708.9	23.4	1	41000.0*
29.7	2	55.4	27.6	2	432.2	25.5	8	4908.9	23.4	1	41000.0*

Censored observations are indicated with an asterisk *.

Failure Time in hours of Kevlar 49 Wrapped Pressure Vessels

29.7	7	61.2	27.6	1	453.4	25.5	2	5556.0	23.4	4	41000.0*
29.7	5	87.5	27.6	2	514.1	25.5	6	6271.1	23.4	4	41000.0*
29.7	8	98.2	27.6	6	514.2	25.5	8	7332.0	23.4	4	41000.0*
29.7	3	101.0	27.6	6	541.6	25.5	8	7918.7	23.4	4	41000.0*
29.7	2	111.4	27.6	2	544.9	25.5	6	7996.0	23.4	8	41000.0*
29.7	6	144.0	27.6	8	554.2	25.5	8	9240.3	23.4	8	41000.0*
29.7	2	158.7	27.6	1	664.5	25.5	8	9973.0	23.4	8	41000.0*

Gerstle, F.P.; and Kunz, S.C (1983). "Prediction of Long-term Failure in Kevlar 49 Composites" in *Long-term Behavior of Composites*, ASTM STP 813, T. K. O'Brian Ed., American Society for Testing and Materials, pp. 263-92, Philadelphia.

Fixed Spool Effects Model:

$$F(t) = P(T \le t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] = \Phi_{sev}\left(\frac{\log t - \mu}{\sigma}\right),$$

$$t \ge 0, \;\; \beta > 0, \;\; \eta > 0, \;\; \mu = \log\ \eta, \;\; \sigma = \frac{1}{\beta}$$

$$\mu = \log(\eta) = \beta_0 + \beta_1 \log(s) + \psi_k, \;\; \mathrm{k} = 1, \ldots 8$$

$$\psi_8 = 0$$

SAS dummy variables

Frequentist Maximum Likelihood Results Reported by Crowder et al. (1991) for Fixed Spool Effect Model.

Point estimates for
1st percentile failure time at (a) 23.4 MPa;
50th percentile failure time at (b) 22.5 MPa

Spool	**(a) F-Time (hours)**	**Lower CL**	**Upper CL**	**(b) F-Time (x 1000 hours)**	**Lower CL**	**Upper CL**
All	70	22	225	88	42	187
1	3762	1701	8317	263	138	502
2	461	222	957	32.2	19.3	54.0
3	217	95	497	15.2	8.15	28.4
4	6264	2757	14234	438	221	869
5	874	369	2070	61.1	32.0	117
6	709	322	1563	49.6	28.2	87.4
7	131	56	305	9.19	4.72	17.9
8	2108	970	4581	147	79.6	273

"All" refers to a model where no spool effect is considered.
CL stands for **95% confidence limits** interpreted according to the frequentist paradigm.

Statistical Analysis of Reliability Data by Crowder, Kimber, Smith, and Sweeting

They remark that in the case of problem (a) it is noteworthy that all the point and lower and upper confidence limits for the separate spools are much greater than the corresponding quantities for all the spools combined. The explanation is that is that the model with no spool effects leads to an estimate of the Weibull shape parameter ($\beta = 0.68$) much smaller than the estimate for the fixed spool effect model ($\beta = 1.26$). Hence, the estimated percentiles are much lower in the lower tail and much higher in the upper tail. The lesson is that ignoring a vital parameter, such as spool effect, may not only be significant in itself but may also lead to bias in estimating the other parameters.

Independent Priors for the Fixed Effects Model

$$\beta_0 \sim N(0, 0.001), \quad \beta_1 \sim N(0, 0.001),$$
$$\beta \sim Gamma(1, 0.2),$$
$$\psi_k \sim N(0, 0.001), \text{ k} = 1, \dots, 7$$

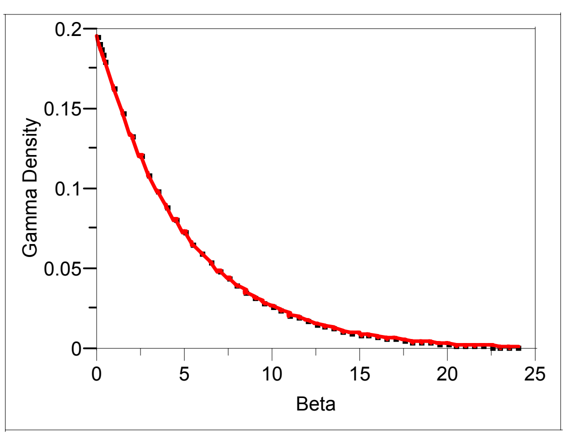
The normal priors are parameterized with the mean and the precision which is the inverse of the variance. Independence is assumed.

The $Gamma(\alpha, \beta)$ distribution has the following density function and parameterization:

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \theta \geq 0$$

$$\alpha = Shape \;\; parameter > 0, \quad \beta = Inverse \;\; scale \;\; parameter > 0$$

Density of Gamma (1, 0.2) prior



Percent (%)	1	5	10	25	40	50	75	99
Percentile for Beta	0.05	0.26	0.53	1.4	2.6	3.5	6.9	23

Bayesian Fit of the Fixed Spool Effect Model with Gamma (1, 0.2) Prior for Beta

Point estimates for
1st percentile failure time at (a) 23.4 MPa;
50th percentile failure time at (b) 22.5 MPa

Spool	**(a) F-Time (hours)**	**Lower CL**	**Upper CL**	**(b) F-Time (x 1000 hours)**	**Lower CL**	**Upper CL**
All	62.32	17.38	177.1	73.57	40.88	135.9
1	3051	1249	6665	248.8	138.2	488.6
2	364.6	155.2	738.2	29.8	18.48	49.7
3	174	67.75	391.8	14.2	7.893	27.28
4	5015	2003	11200	409.3	219.7	825.4
5	715.7	268.6	1697	58.3	31.35	118.9
6	572.4	229.2	1243	46.7	27.22	84.9
7	104.4	40.25	238.7	8.519	4.595	16.89
8	1715	711.9	3686	140	79.94	267.2

Random Spool Effects Model:

$$F(t) = P(T \le t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] = \Phi\left(\frac{\log t - \mu}{\sigma}\right),$$

$$t \ge 0,\ \ \beta > 0,\ \ \eta > 0,\ \mu = \log \eta,\ \sigma = 1/\beta$$

$$\mu = \log(\eta) = \beta_0 + \beta_1 \log(s) + \psi_k,\ k = 1,\ldots,8$$

$$\psi_k \sim N\left(0,\ 1/\sigma^2\right),\ \mathrm{k} = 1,\ldots,8$$

The distributions of $\psi_1,\ldots,\psi_8$ are assumed to have independent normal distributions with variance σ^2. We use the following independent vague priors:

$$\beta_0 \sim N(0,0.001),\ \beta_1 \sim N(0,0.001),\ \beta \sim Gamma(1,0.2)$$

$$\tau = \left(\sigma^2\right)^{-1} \sim Gamma(0.001,0.001)$$

Bayesian Fit of Random Spool Effect Model with Gamma (1, 0.2) Prior

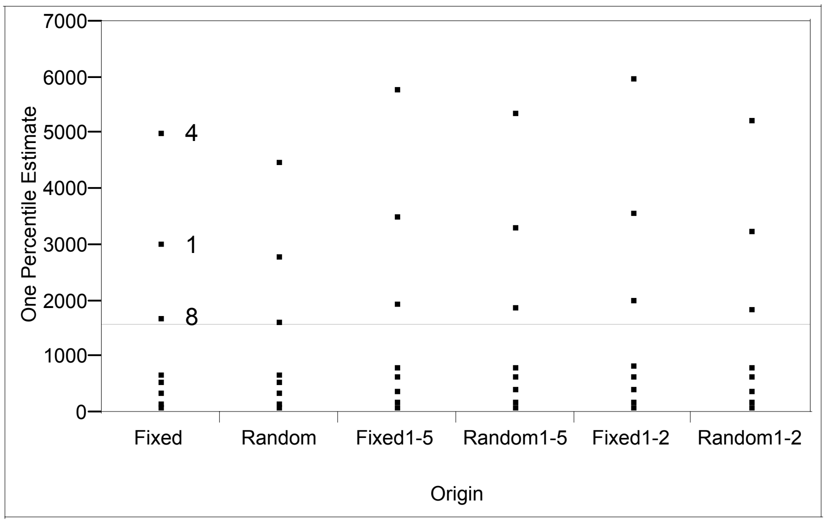
Point estimates for
1st percentile failure time at (a) 23.4 MPa;
50th percentile failure time at (b) 22.5 MPa

Spool	**(a) F-Time (hours)**	**Lower CL**	**Upper CL**	**(b) F-Time (x 1000 hours)**	**Lower CL**	**Upper CL**
All	62.32	17.38	177.1	73.57	40.88	135.9
1	2819	1144	6117	221.5	121.1	414.6
2	362.4	153.2	732.6	28.56	17.14	46.42
3	179.5	70.06	402.3	14.08	7.616	26.86
4	4524	1773	10060	356.6	185.8	682.8
5	708.9	267.9	1657	55.42	29.92	110
6	570.5	229.3	1228	44.72	25.76	79.81
7	108.8	42.04	247.7	8.547	4.469	16.8
8	1635	675.2	3497	128.4	72.12	237.3
Random spool	671	21.96	19290	53.68	1.867	1479

Remarks

- Random effects model when compared to the fixed effect model:
  - Shrinks extreme estimates towards a central value
  - Credibility intervals associated with extreme estimates are narrower
  - Inference possible for a random spool selected from the population of spools

Shrinkage of Random Effect Model Estimates as Compared to Fixed Effect Model Estimates



Bayesian Fit of Random Spool Effect Model with Gamma (1, 0.2) Prior: Prediction Intervals

(a) 23.4 MPa; (b) 22.5 MPa

Spool	(a) F-Time (hours)	Lower CL	Upper CL	(b) F-Time (x 1000 hours)	Lower CL	Upper CL
All	29650	220	369600	73.490	0.5496	942.300
1	90820	5560	421900	218.90	13.37	1036.00
2	11750	734	51270	28.21	1.73	125.80
3	5802	354	26960	13.95	0.84	66.16
4	146200	8906	685200	350.80	21.07	1680.00
5	22950	1374	108100	55.04	3.31	264.20
6	18520	1138	83520	44.36	2.71	204.40
7	3517	211	16580	8.44	0.51	40.79
8	52910	3251	244400	126.90	7.91	598.00
Random spool	19850.00	302.70	793000	47.95	0.73	1917.00

Bayesian Fit of Random Spool Effect Model with Gamma (1, 0.2) Prior: Probability of Failure by 1000 hours at 23.4 MPa

Spool	Probability of Failure (%)	Lower CL (%)	Upper CL(%)
All	0.0650%	0.0378%	0.1071%
1	0.2850%	0.0847%	0.8675%
2	3.3870%	1.5270%	7.0920%
3	7.7910%	3.3340%	16.3500%
4	0.1606%	0.0427%	0.5454%
5	1.5160%	0.5065%	4.0220%
6	1.9680%	0.7568%	4.7040%
7	13.8500%	6.1620%	27.5100%
8	0.5514%	0.1825%	1.5150%
Random Spool	1.6140%	0.0246%	60.8300%

Bayesian Fit of Random Spool Effect Model with Gamma (1, 0.2) Prior: Probability of Failure by 1000 hours at 22.5 MPa

Spool	Probability of Failure (%)	Lower CL (%)	Upper CL(%)
All	0.0355%	0.0184%	0.0650%
1	0.0981%	0.0246%	0.3515%
2	1.1780%	0.4467%	2.9240%
3	2.7460%	1.0150%	6.7810%
4	0.0553%	0.0123%	0.2219%
5	0.5231%	0.1509%	1.6160%
6	0.6806%	0.2234%	1.9080%
7	4.9860%	1.9210%	11.7800%
8	0.1899%	0.0534%	0.6126%
Random Spool	0.5612%	0.0080%	27.9600%

Conclusions

- Bayesian methods enable us to answer the questions that are of interest to practitioners with a random effects model
  - Quantiles
  - Predictions for new unit's times to failure
  - Probabilities of failure at a given time
  - No asymptotic approximations
- No handling random effects correctly can lead to misleading results
- Eight spools are not enough to ascertain the reliability of the Space Shuttle pressure vessel system

Feiveson, A. H.; and Kulkarni, P. M. (2000). "Reliability of Space-Shuttle Pressure Vessels with Random Batch Effects". *Technometrics* 42, pp. 332-344.