Unit 10: Planning Life Tests

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Unit 10 Objectives

• Explain the basic ideas behind planning a life test
• Use simulation to anticipate the results, analysis, and precision for a proposed test plan
• Explain large-sample approximate methods to assess precision of future results from a reliability study
• Compute sample size needed to achieve a degree of precision
• Assess tradeoffs between sample size and length of a study.
• Illustrate the use of simulation to calibrate the easier-to-use large-sample approximate methods
Basic Ideas in Test Planning

• The enormous cost of reliability studies makes it essential to do careful planning. Frequently asked questions include:
  – How many units do I need to test in order to estimate the .1 quantile of life?
  – How long do I need to run the life test?
• Clearly, more test units and more time will buy more information and thus more precision in estimation
• To anticipate the results from a test plan and to respond to the questions above, it is necessary to have some **planning** information about the life distribution to be estimated

Engineering Planning Values and Assumed Distribution for Planning an Insulation Life Test

Want to estimate \( r_{.1} \) of the life distribution of a newly developed insulation. Tests are run at higher than usual volt/thickness to cause failures to occur more quickly.

Information (planning values) from engineering

• Expect about 20% failures in the 1000 hour test and about 12% failures in the first 500 hours of the test.

• Willing to assume a Weibull distribution to describe failure-time.

• Equivalent information for **planning values**: \( \eta^\text{0} = 6464 \) hours (or \( \mu^\text{0} = \log(6464) = 8.774 \), \( \beta^\text{0} = .8037 \) (or \( \sigma^\text{0} = 1/\beta^\text{0} = 1.244 \)).

**Starting point**: Use simulated data to assess precision.
Simulation as a Tool for Test Planning

- Use assumed model and planning values of model parameters to simulate data from the proposed study
- Analyze the data perhaps under different assumed models
- Assess precision provided
- Simulate many times to assess actual sample-to-sample differences
- Repeat with different sample sizes to gauge needs
- Repeat with different input planning values to assess sensitivity to these inputs.

Any surprises?
ML Estimates from 50 Simulated Samples of Size
\[ n = 20, \ t_c = 400 \] from a Weibull Distribution
with \[ \mu^\circ = 8.774 \] and \[ \sigma^\circ = 1.244 \]

ML Estimates from 50 Simulated Samples of Size
\[ n = 80, \ t_c = 400 \] from a Weibull Distribution
with \[ \mu^\circ = 8.774 \] and \[ \sigma^\circ = 1.244 \]
ML Estimates from 50 Simulated Samples of Size

\[ n = 20, \ t_c = 1000 \] from a Weibull Distribution

with \( \mu^D = 8.774 \) and \( \sigma^D = 1.244 \)

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ML Estimates from 50 Simulated Samples of Size

\[ n = 80, \ t_c = 1000 \] from a Weibull Distribution

with \( \mu^D = 8.774 \) and \( \sigma^D = 1.244 \)
Simulations of Insulation Life Tests

- ML estimates obtained from 50 simulated samples of size \( n = 20, 80 \), from a Weibull distribution with \( \mu = 8.774, \sigma = 1.244 (\beta = .8037) \).

- The vertical lines at \( t_c = 400, 1000 \) hours (shown with the thicker line) indicates the censoring time (end of the test).

- The horizontal line is drawn at \( p = .1 \) so to provide a better visualization of the distribution of estimates of \( t_c \).

- Results at \( t_c = 400 \) and \( n = 20 \) are highly variable.

Trade-offs Between Test Length and Sample Size

Geometric average \( \hat{R} \) factor from 50 simulated exponential samples (\( \theta = 5 \)) for combinations of sample size \( n \) and test length \( t_c \) (conditional on \( r \geq 1 \) failures)

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<tr>
<th>Test Length ( t_c )</th>
<th>Sample Size ( n )</th>
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<td>(16)</td>
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Numbers within parenthesis are the expected number of failures at each test condition.
Assessing the Variability of the Estimates

- For positive quantile \( t_p \), an approximate \( 100(1 - \alpha)\% \) confidence interval is given by
  \[
  [t_p, \hat{t}_p] = [\bar{t}_p/\hat{R}, \hat{t}_p\hat{R}]
  \]
  where \( \hat{R} = \exp \left[ \frac{z_{1-(\alpha/2)}\hat{\sigma}}{\hat{\theta}^{1/2}\hat{\gamma}} \right] \). The factor \( \hat{R} > 1 \) is an indication of the width of the interval and can be used to assess the variability in the estimates \( \hat{t}_p \).

- For an unrestricted quantile \( y_p \), an approximate \( 100(1 - \alpha)\% \) confidence interval is given by
  \[
  [y_p, \bar{y}_p] = [\bar{y}_p - \hat{D}, \bar{y}_p + \hat{D}]
  \]
  where \( \hat{D} = z_{1-(\alpha/2)}\hat{\sigma}/\hat{\theta}^{1/2}\hat{\gamma} \). The half-width \( \hat{D} \) is an indication of the width of the interval and can be used to assess the variability in the estimates \( \bar{y}_p \).

Simulations of Insulation Life Test-Continued

Some important points about the effect that sample size will have on our ability to make inferences:

- For the \( t_c = 400 \) and \( n = 5 \) simulation
  - Enormous amount of variability in the ML estimates.
  - For several of the simulated data sets, no ML estimates exist because all units were censored.

- Increasing the experiment length to \( t_c = 1000 \) and the sample size to \( n = 80 \) provides
  - A more stable estimation process.
  - A substantial improvement in precision.
Motivation for Use of Large-Sample Approximations of Test Plan Properties

Asymptotic methods provide:

- Simple expressions giving precision of a specified estimator as a function of sample size
- Simple expressions giving needed sample size as a function of specified precision of a specified estimator
- Simple tables and graphs that will allow easy assessments of tradeoffs in test planning decisions like sample size and test length
- Can be fine tuned with simulation evaluation
Asymptotic Variances

Under certain regularity conditions the following results hold asymptotically (large sample)

- $\hat{\theta} \sim \text{MVN}(\theta, \Sigma_{\theta})$, where $\Sigma_{\theta} = I_{\theta}^{-1}$, and
  
  $I_\theta = \mathbb{E} \left[ -\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta \partial \theta'} \right] = \sum_{i=1}^{n} \mathbb{E} \left[ -\frac{\partial^2 \mathcal{L}_i(\theta)}{\partial \theta \partial \theta'} \right].$

- For a scalar $g = g(\hat{\theta}) \sim N[g(\theta), \text{Avar}(\hat{g})]$, where
  
  $\text{Avar}(\hat{g}) = \left[ \frac{\partial g(\theta)}{\partial \theta} \right]^{T} \Sigma_{\theta} \left[ \frac{\partial g(\theta)}{\partial \theta} \right].$

- When $g(\theta)$ is positive for all $\theta$, then
  
  $\log[g(\theta)] \sim \text{Normal} \{ \log[g(\theta)], \text{Avar}[\log(\hat{g})] \}$, where
  
  $\text{Avar}[\log(\hat{g})] = \left( \frac{1}{\hat{g}} \right)^2 \text{Avar}(\hat{g}).$

---

Delta Method for Two Parameters

$$
\begin{bmatrix}
\frac{\partial g(\theta)}{\partial \theta_1} \\
\frac{\partial g(\theta)}{\partial \theta_2}
\end{bmatrix} v_1 v_2 =

\begin{bmatrix}
\frac{\partial g(\theta)}{\partial \theta_1} \\
\frac{\partial g(\theta)}{\partial \theta_2}
\end{bmatrix} v_1 v_2 + \begin{bmatrix}
\frac{\partial g(\theta)}{\partial \theta_1} \\
\frac{\partial g(\theta)}{\partial \theta_2}
\end{bmatrix} v_1 v_2

\begin{bmatrix}
\frac{\partial g(\theta)}{\partial \theta_1} \\
\frac{\partial g(\theta)}{\partial \theta_2}
\end{bmatrix}

\begin{bmatrix}
\frac{\partial^2 g(\theta)}{\partial \theta_1 \partial \theta_1} v_1^2 + 2 \left( \frac{\partial g(\theta)}{\partial \theta_1} \frac{\partial g(\theta)}{\partial \theta_2} \right) v_1 v_2 + \left( \frac{\partial g(\theta)}{\partial \theta_2} \right)^2 v_2^2
\end{bmatrix}
$$
Asymptotic Approximate Standard Errors for a Function of the Parameters $g(\theta)$

Given an assumed model, parameter values (but not sample size), one can compute scaled asymptotic variances.

- The variance factors $V_{\hat{g}} = n\text{Var}(\hat{g})$ and $V_{\log(\hat{g})} = n\text{Var}[\log(\hat{g})]$ may depend on the actual value of $\theta$ but they do not depend on $n$.

To compute these variance factors one uses planning values for $\theta$ (denoted by $\Theta^0$) as discussed later.

- The asymptotic standard error for $\hat{g}$ and $\log(\hat{g})$ are

$$Ase(\hat{g}) = \frac{1}{\sqrt{n}} \sqrt{V_{\hat{g}}}$$

$$Ase[\log(\hat{g})] = \frac{1}{\sqrt{n}} \sqrt{V_{\log(\hat{g})}}.$$ 

- Easy to choose $n$ to control $Ase$.

Example

Example 10.3  Sample Size Needed to Estimate the Mean of Light Bulb Life.

The life of some types of incandescent light bulbs can be modeled adequately with a normal distribution. Depending on the particular design, mean life might be on the order of 1000 hours with a standard deviation under 200 hours. To satisfy a request from marketing, it was desired to plan a life test that would estimate mean life of light bulbs so that a 95% confidence interval has a half-width that is approximately 30 hours. The product engineers are willing to assume that life is adequately described by a normal distribution with a standard deviation no larger than $\sigma^0 = 200$ hours and there is enough time to let all of the bulbs fail before analyzing the data.

From elementary statistics, $\hat{\mu} = \bar{x}$ so $V_{\hat{\mu}} = \text{Var}(\bar{x}) = \sigma^2$ and $V_{\bar{x}} = (\sigma^2)^2 = (200)^2$. Substituting this and $D_T = 30$ into (10.4) shows that the number of bulbs needed is

$$n = \frac{\bar{x}^2(1-n/2)}{V_{\bar{x}} D_T^2} = \frac{(1.96)^2(200)^2}{30^2} \approx 171.$$
Sample Size Determination for Positive Functions of the Parameters

- When \( g(\theta) > 0 \) for all \( \theta \), an approximate 100(1 - \( \alpha \))% confidence interval for \( \log(g(\theta)) \) is

\[
\left[ \log(\tilde{g}), \log(\bar{g}) \right] = \log(\tilde{g}) \pm (1/\sqrt{n})z_{(1-\alpha/2)}\sqrt{\hat{V}_{\log(\tilde{g})}} = \log(\tilde{g}) \pm \log(R).
\]

Exponentiation yields a confidence interval for \( g \)

\[
[\tilde{g}, \bar{g}] = [\tilde{g}/R, \bar{g}R]
\]

\[
R = \exp \left[ (1/\sqrt{n})z_{(1-\alpha/2)}\sqrt{\hat{V}_{\log(\tilde{g})}} \right] = \bar{g}/\tilde{g} = \bar{g}/2 = \sqrt{2}/\tilde{g}.
\]

- Replace \( \hat{V}_{\log(\tilde{g})} \) with \( \hat{V}_{\log(\tilde{g})}^0 \) and solve for \( n \) to compute the needed sample size giving

\[
n = \frac{\hat{V}_{\log(\tilde{g})}^0}{(z_{(1-\alpha/2)})^2}.
\]

Sample Size Determination for Positive Functions of the Parameters-Continued

Test plans with a sample size of

\[
n = \frac{\hat{V}_{\log(\tilde{g})}^0}{(z_{(1-\alpha/2)})^2}.
\]

provides confidence intervals for \( g(\theta) \) with the following characteristics:

- In repeated samples approximately 100(1 - \( \alpha \))% of the intervals will contain \( g(\theta) \).

- In repeated samples \( \hat{V}_{\log(\tilde{g})} \) is random and if \( \hat{V}_{\log(\tilde{g})} > \hat{V}_{\log(\tilde{g})}^0 \) then the ratio \( R = \bar{g}/\tilde{g} \) will be greater than \( |R_T|^2 \).

- The ratio \( R = \bar{g}/\tilde{g} \) will be greater than \( |R_T|^2 \) with a probability of order .5.
Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life

- Need a test plan that will estimate the mean life of insulation specimens at highly-accelerated (i.e., higher than usual voltage to get failure information quickly) conditions.

- Desire a 95% confidence interval with endpoints that are approximately 50% away from the estimated mean (so $R_T = 1.5$).

- Can assume an exponential distribution with a mean $\theta^D = 1000$ hours.

- Simultaneous testing of all units; must terminate test at 500 hours.

Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life - Continued

- ML estimate of the exponential mean is $\hat{\theta} = \frac{TTT}{r}$, where $TTT$ is the total time on test and $r$ is the number of failures. It follows that

$$V_{\hat{\theta}} = n \text{Var}(\hat{\theta}) = n \left[ -\frac{\partial^2 \ell(\theta)}{\partial \theta^2} \right] = \frac{\theta^2}{1 - \exp \left(-\frac{1}{\theta} \right)}$$

from which

$$V_{\log(\hat{\theta})} = \frac{V_{\hat{\theta}}}{[\theta^D]^2} = \frac{1}{1 - \exp \left(-\frac{500}{1000} \right)} = 2.5415.$$ 

Thus the number of needed specimens is

$$n = \frac{\frac{1}{1 - \alpha/2} \cdot V_{\log(\hat{\theta})}}{[\log(R)]^2} = \frac{(1.96)^2 \cdot 2.5415}{[\log(1.5)]^2} \approx 60.$$
Derivation.1

\[ L = \prod_{i=1}^{r} \left( \frac{e^{-\frac{t_i}{\theta}}}{\theta} \right) \prod_{i=r+1}^{n} e^{-\frac{t_i}{\theta}} = \theta^{-r} e^{\frac{TTT}{\theta}} \]

⇒

\[ l = -r \log \theta - TTT \theta^{-1} \]

⇒

\[ \frac{\partial l}{\partial \theta} = -r \theta^{-1} + TTT \theta^{-2} \]

⇒

\[ \frac{\partial^2 l}{\partial \theta^2} = r \theta^{-2} - 2TTT \theta^{-3} \]

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Derivation.2

\[ E \left[ -\frac{\partial^2 l}{\partial \theta^2} \right] = E \left[ 2TTT \theta^{-3} - R \theta^{-2} \right] \]

\[ = 2\theta^{-3} \left( E[TTT] \right) - \theta^{-2} E[R] \]

\[ = 2\theta^{-3} \left( \theta E[R] \right) - \theta^{-2} E[R] \]

\[ = \theta^{-2} \left( E[R] \right) \]

\[ = \theta^{-2} \left( n \times P[T \leq t_c] \right) = \theta^{-2} (np_c) \]

\[ = n\theta^{-2} \left( 1 - e^{-\frac{t_c}{\theta}} \right) \]

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Location-Scale Distributions and Single Right Censoring

Asymptotic Variance-Covariance

Here we specialize the computation of sample sizes to situations in which

- \( \log(T) \) is location-scale \( \Phi \) with parameters \((\mu, \sigma)\).

- When the data are Type I singly right censored at \( t_c \). In this case,

\[
\frac{n}{\sigma^2} \Sigma(\hat{\mu}, \hat{\sigma}) = \frac{1}{\sigma^2} \begin{bmatrix}
V_{\hat{\mu}} & V_{\hat{\mu}, \hat{\sigma}} \\
V_{\hat{\mu}, \hat{\sigma}} & V_{\hat{\sigma}}
\end{bmatrix} = \left[ \frac{\sigma^2}{n} I(\mu, \sigma) \right]^{-1} = \begin{bmatrix}
f_{11} & f_{12} \\
f_{12} & f_{22}
\end{bmatrix}^{-1}
\]

\[
= \left( \frac{1}{f_{11} f_{22} - f_{12}^2} \right) \begin{bmatrix}
f_{22} & -f_{12} \\
-f_{12} & f_{11}
\end{bmatrix}
\]

where the \( f_{ij} \) values depend only on \( \Phi \) and the standardized censoring time \( \zeta_c = [\log(t_c) - \mu] / \sigma \) [or equivalently, the proportion failing by \( t_c \), \( \Phi(\zeta_c) \)].

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Location-Scale Distributions and Single Right Censoring

Fisher Information Elements

The \( f_{ij} \) values are defined as:

\[
f_{11} = f_{11}(\zeta_c) = \frac{\sigma^2}{n} \mathbb{E} \left[ -\frac{\partial^2 \mathcal{L}_n(\mu, \sigma)}{\partial \mu^2} \right]
\]

\[
f_{22} = f_{22}(\zeta_c) = \frac{\sigma^2}{n} \mathbb{E} \left[ -\frac{\partial^2 \mathcal{L}_n(\mu, \sigma)}{\partial \sigma^2} \right]
\]

\[
f_{12} = f_{12}(\zeta_c) = \frac{\sigma^2}{n} \mathbb{E} \left[ -\frac{\partial^2 \mathcal{L}_n(\mu, \sigma)}{\partial \mu \partial \sigma} \right]
\]

The \( f_{ij} \) values are available from tables or algorithm LSINF for the SEV (Weibull), normal (lognormal), and logistic (loglogistic) distributions.

For a single fixed censoring time, the asymptotic variance-covariance factors \( \frac{1}{\sigma^2} V_{\hat{\mu}}, \frac{1}{\sigma^2} V_{\hat{\sigma}}, \) and \( \frac{1}{\sigma^2} V_{\hat{\mu}, \hat{\sigma}} \) are easily tabulated as a function of \( \zeta_c \).

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Table of Information Matrix Elements and Variance Factors

Table C.20 provides for the normal/lognormal distributions, as functions of the standardized censoring time $\zeta$: 

- $100\Phi(\zeta)$, the percentage in the population failing by the standardized censoring time.
- Fisher information matrix elements $f_{11}, f_{22},$ and $f_{12}$.
- The asymptotic variance-covariance factors $\frac{1}{\sigma^2}V(\hat{\mu}), \frac{1}{\sigma^2}V(\hat{\sigma}),$ and $\frac{1}{\sigma^2}V(\hat{\mu}, \hat{\sigma})$.
- Asymptotic correlation $\rho(\hat{\mu}, \hat{\sigma})$ between $\hat{\mu}$ and $\hat{\sigma}$.
- The $\sigma$-known asymptotic variance factor $\frac{1}{\sigma^2}V_{\hat{\mu}, \sigma} = n\text{AVar}(\hat{\mu})$, and the $\mu$-known factor $\frac{1}{\sigma^2}V_{\hat{\sigma}, \mu} = n\text{AVar}(\hat{\sigma})$.

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<th>$f_{22}$</th>
<th>$f_{12}$</th>
<th>$\frac{1}{\sigma^2}V(\hat{\mu})$</th>
<th>$\frac{1}{\sigma^2}V(\hat{\sigma})$</th>
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Large-Sample Asymptotic Variance for Estimators of Functions of Location-Scale Parameters

It is straightforward to compute asymptotic variance factors for functions of parameters. For example, when $\tilde{g} = g(\tilde{\mu}, \tilde{\sigma})$

$$A\text{var}(\tilde{g}) = \left[ \frac{\partial g}{\partial \mu} \right]^2 A\text{var}(\tilde{\mu}) + \left[ \frac{\partial g}{\partial \sigma} \right]^2 A\text{var}(\tilde{\sigma}) + 2 \left[ \frac{\partial g}{\partial \mu} \right] \left[ \frac{\partial g}{\partial \sigma} \right] A\text{cov}(\tilde{\mu}, \tilde{\sigma})$$

Thus

$$V_{\tilde{g}} = \left( \frac{1}{g} \right)^2 V_{\tilde{g}} = \exp(2g) V_{\tilde{g}}$$

Sample Size to Estimate a Quantile of $T$

when $\log(T)$ is Location-Scale $(\mu, \sigma)$

- Let $g(\theta) = t_p$ be the $p$ quantile of $T$. Then $\log(t_p) = \mu + \Phi^{-1}(p)\sigma$, where $\Phi^{-1}(p)$ is the $p$ quantile of the standardized random variable $Z = [\log(T) - \mu]/\sigma$.

- From the previous results, $n$ is given by

$$n = \frac{z^2}{\log(\log(t_p))^2}$$

where $V_{\log(t_p)}$ is obtained by evaluating

$$V_{\log(t_p)} = \left\{ V_{\mu} + \left[ \Phi^{-1}(p) \right]^2 V_{\sigma} + 2 \left[ \Phi^{-1}(p) \right] V_{\tilde{g} \tilde{\sigma}} \right\}$$

at $\theta^\alpha = (\mu^\alpha, \sigma^\alpha), \zeta^\alpha = [\log(t_c) - \mu]^\alpha/\sigma^\alpha$.

- Figure 10.5 gives $V_{\log(t_p)}$ as a function of $p_c = \Pr(Z \leq \zeta_c)$ for the Weibull distribution. To obtain $n$ one also needs to specify $\Phi$ and a target value $R_T$ for $R = \tilde{g}/\tilde{\sigma} = g/\sigma = \sqrt{g/\sigma}$. 

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Sample Size Needed to Estimate $t_{.1}$ of a Weibull Distribution Used to Describe Insulation Life

- Again expect about 20% failures in the 1000 hour test and 12% failures in the first 500 hours. Equivalent information: $\mu^D = 8.774$, $\sigma^D = 1.244$ (or $\beta^D = 1/1.244 = .8037$).

- Need a test plan that will estimate the Weibull .1 quantile (so $p = .1$) such that a 95% confidence interval will have endpoints that are approximately 50% away from the estimated mean (so $R_f = 1.5$). For a 1000-hour test, $p_c = .2$.

- By computing from tables and formula or from Figure 10.5, $\frac{1}{\sigma^D} V_{\log}(t_p) = 7.28$ so $V_{\log}(t_p) = 7.28 \times (1.244)^2 = 11.266$.

Thus, 
$$n = \frac{Z_{1-\alpha/2}V^D_{\log}(t_{.1})}{[\log(R)]^2} = \frac{(1.96)^2(11.266)}{[\log(1.5)]^2} \approx 263.$$
Variance Factor $\frac{1}{\sigma^2} V_{\log(t_p)}$ for ML Estimation of
Weibull Distribution Quantiles as a Function of $p_c$, the
Population Proportion Failing by Time $t_c$ and $p$, the
Quantile of Interest (Figure 10.5)

Variance Factor $\frac{1}{\sigma^2} V_{\log(t_p)}$ for ML Estimation of
Lognormal Distribution Quantiles as a Function of $p_c$,
the Population Proportion Failing by Time $t_c$ and $p$,
the Quantile of Interest (Figure 10.6)
Figures for Sample Sizes to Estimate Weibull, Lognormal, and Loglogistic Quantiles

Figures give plots of the factor \( \frac{1}{\sigma^2} V_{\log(q_p)} \) for quantile of interest \( p \) as a function of \( p = \Pr(Z \leq z_\alpha) \) for the Weibull, lognormal, and loglogistic distributions. Close inspection of the plots indicates the following:

- Increasing the length of a life test (increasing the expected proportion of failures) will always reduce the asymptotic variance. After a point, however, the returns are diminishing.

- Estimating quantiles with \( p \) large or \( p \) small generally results in larger asymptotic variances than quantiles near to the expected proportion failing.

Generalization: Location-Scale Parameters and Multiple Censoring

In some applications, a life test may run in parts, each part having a different censoring time (e.g., testing at two different locations or beginning as lots of units to be tested are received). In this case we need to generalize the single-censoring formula. Assume that a proportion \( \delta_i (\sum_{i=1}^{d} \delta_i = 1) \) of data are to be run until right censoring time \( t_i \) or failure (which ever comes first). In this case,

\[
\frac{n}{\sigma^2} \sum \delta_i \left( \frac{V_{\mu}}{V_{\mu}^*} \right) \left( \frac{V_{\sigma}}{V_{\sigma}^*} \right) = \left[ \frac{1}{n} \mathbb{I}_{(\mu, \sigma)} \right]^{-1} \\
= \left( \begin{array}{cc} \frac{1}{J_{11} J_{22} - J_{12} J_{12}} & -J_{12} \\
J_{12} & J_{11} \end{array} \right)
\]

where \( J_{11} = \sum_{i=1}^{d} \delta_i f_{i1}(z_i) \), \( J_{22} = \sum_{i=1}^{d} \delta_i f_{i2}(z_i) \), and \( J_{12} = \sum_{i=1}^{d} \delta_i f_{i12}(z_i) \) where \( z_i = (\log(t_i) - \mu)/\sigma \).

In this case, the asymptotic variance-covariance factors \( \frac{1}{\sigma^2} V_{\mu}^*, \frac{1}{\sigma^2} V_{\sigma}^* \), and \( \frac{1}{\sigma^2} V_{(\mu, \sigma)}^* \) depend on \( \Phi \), the standardized censoring times \( z_i \), and the proportions \( \delta_i, i = 1, \ldots, k \).
Test Plans to Demonstrate Conformity with a Reliability Standard

Objective: to find a sample size to demonstrate with some level of confidence that reliability exceeds a given standard.

- The reliability is specified in terms of a quantile, say \( t_p \).

\[
t_p > t^1_p
\]

where \( t^1_p \) is a specified value.

For example, for a component to be installed in a system with a 1-year warranty, a vendor may have to demonstrate that \( t_{0.01} \) exceeds \( 24 \times 365 = 8760 \) hours.
- Equivalently, in terms of failure probabilities the reliability requirement could be specified as

\[
F(t_c) < p^1.
\]

For the example, \( t_c = 8760 \) and \( p^1 = .01 \).

Minimum Sample Size
Reliability Demonstration Test Plans

- In general the demonstration that \( t_p > t^1_p \) is successful at the \( 100(1 - \alpha)\% \) level of confidence if \( t_p > t^1_p \).
- Suppose that failure-times are Weibull with a given \( \beta \). A minimum sample size test plan is one that has a particular sample size \( n \) (depending on \( \beta, \alpha, p \) and amount of time available for testing).
- The minimum sample size test plan is: Test \( n \) units until \( t_c \) where \( n \) is the smallest integer greater than

\[
\frac{1}{k^\beta} \times \log(\alpha) \frac{1}{\log(1 - p)}
\]

and \( k = t_c/t^1_p \).
- If there is zero failures during the test the demonstration is successful.
Justification for the Weibull Zero-Failures Test Plan

Suppose that failure-times are Weibull with a given $\beta$ and zero failures during a test in which $n$ units are tested until $t_c$. Using the results in Chapter 8, to obtain $100(1-\alpha)\%$ lower bounds for $\eta$ and $t_P$ are

$$\eta = \left[ \frac{2nt_c^\beta}{\chi^2(1-\alpha,2)} \right]^\frac{1}{\beta} \quad \text{and} \quad t_P = \eta \times [-\log(1-p)]^{\frac{1}{\beta}}.$$

- Using the inequality $t_P > t_P^1$, and solving for the smallest integer $n$ such that

$$n \geq \frac{1}{k^\beta} \times \frac{\log(\alpha)}{\log(1-p)}$$

gives the needed minimum sample size, where $k = t_c/t_P^1$. 
Justification for the Weibull Zero-Failures Test Plan (Continued)

- For tests with $k < 1$, which implies extrapolation in time, having a specified value of $\beta$ greater than the true value is conservative (the confidence level is greater than the nominal).

- For tests with $k > 1$ having a specified value of $\beta$ less than the true value is conservative (in the sense that the demonstration is still valid).

- When $k = 1$ the value of $\beta$ does not effect the sample size.

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Additional Comments on Zero Failure Test Plans

- The inequality $t_p > t^\frac{1}{\beta}$ can be solved for $n$, $k$, $\beta$, or $\alpha$. Zero-failure test plans can be obtained for other failure-time distributions with only one unknown parameter.

- Zero-failure test plans can be obtained for any distribution.

- The ideas here can be extended to test plans with one or more failures. Such test plans require more units but provide a higher probability of successful demonstration for a given $t_p > t_p$. 
Other Topics in Chapter 10

- Uncertainty in planning values and sensitivity analysis
- Sample size to estimate unrestricted functions of the parameters, the mean of an exponential, the hazard function of a location-scale distribution
- Test planning for non-location-scale distributions