The Unavailability of a Two-Unit Parallel System with One Traveling Repairperson

Ramón V. León[a], Frank M. Guess, and Kaixiang Tao

***Abstract* — A two-unit parallel system experiences an outage only when both units are down at the same time. We consider the situation when the system undergoes repair by a single repairperson. We assume that the repairperson must travel to the repair site and once the repairperson is at the site no further travel is needed if the second unit goes down while the first one is being repaired. We provide a formula for the unavailabity in this situation assuming that the units' lifetimes are exponential while the travel time and repair time have arbitrary distributions. Our results are different from previous results that simply add travel time to repair time, which overestimates the unavailability. However, the percent overestimation is rather small ranging from less than 0.001% to 1% in most practical applications although the theoretical maximum overestimation could be up to 27%. So it follows that in practice, not very much harm is done by adding travel time to repair time.**

***Index terms* — Unavailability, Two-unit parallel system, Traveling time.**

[a] Author to whom correspondence should be addressed; electronic mail: rleon@utk.edu

NOTATION

$\bar{u}$ = system availability,

u = system unavailability,

λ = processor failure rate

R = random repair time

ER = mean repair time

$L_R(\lambda)$ = L_R = Laplace transform of repair time, R, evaluated at λ

T = random travel time

ET = mean travel time

$L_T(\lambda)$ = L_T = Laplace transform of travel time, T, evaluated at λ

E(T+R) = mean total travel and repair time = ET + ER

I. Introduction

A two-unit parallel system experiences an outage only when both units are down at the same time. Examples of these systems are duplex processors and two-engine airplanes. (The research in this paper was done for a reliability analysis of a computer system with two processors.)

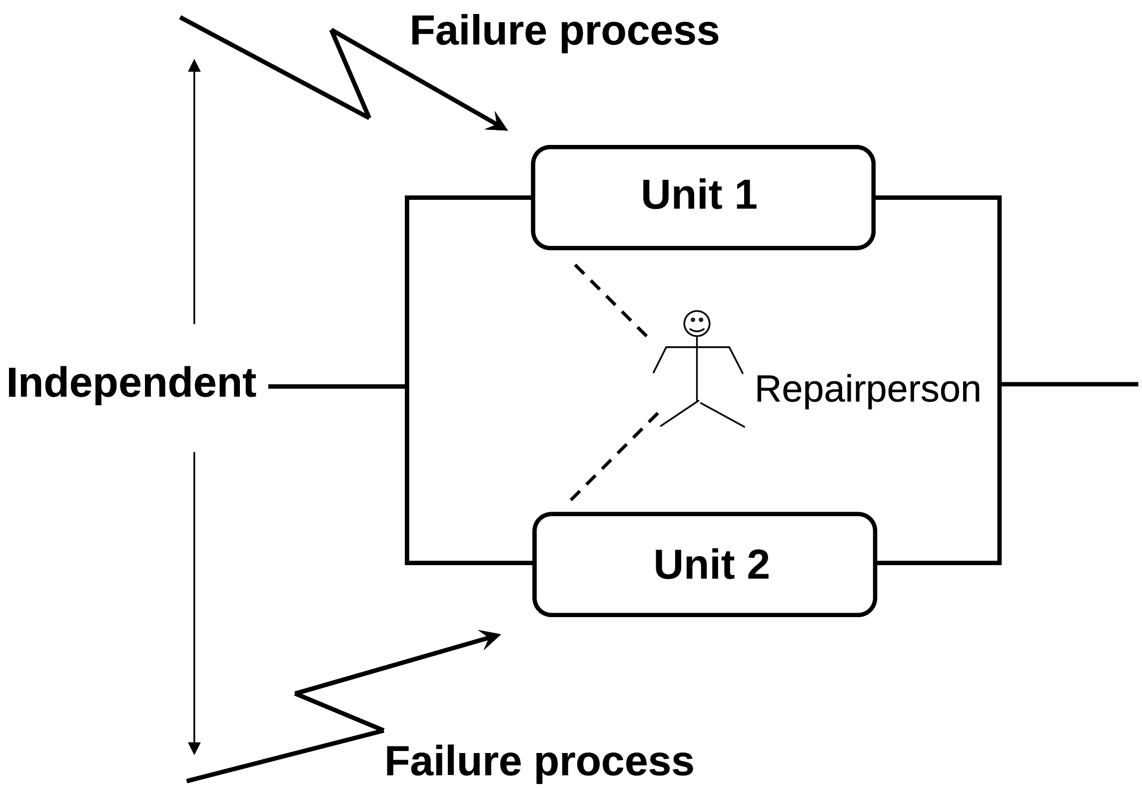

Figure 1. Schematic of a two-unit parallel system attended by a single repairperson.

Gaver systematically studied the reliability properties of a two-unit parallel system attended by a single repairperson [1-2]. After Gaver's pioneering work, the two-unit parallel system received considerable attention during the past several decades. Systems with a cold or warm standby unit attended by a single repairperson or by two repairpersons are studied under different application conditions [3-11]. For instance, Vanderperre analyzed the idle time of two repairmen attending the two-unit parallel system sustained by a cold standby unit [4], Zequeira studied the inspection policy of a two-unit parallel system with failure interaction [7], while Osaki discussed the preventive maintenance policy for a two-unit standby redundant system [8]. However, in all of these studies, the reliability analyses are performed assuming that the repair starts as soon as a failure happens. In various settings, this assumption can be reasonable. In other settings, however, it is not as reasonable an assumption, if not wrong, since the repairperson in some cases has to take some random time to reach the site. Additionally, in other cases, the repairman may wait until a mutually convenient time is available before leaving for the maintenance site. It is intuitive that the length of the traveling time (including the waiting time) will affect the availability or unavailability of the system. Notice that once the repairperson is at the site if a failure occur no further traveling is needed

In this paper, the unavailability of Gaver's parallel system will be revisited. Then, the traveling time will be included in the model and its effects on the unavailability of the system will be discussed quantitatively. Errors bounds are derived when travel time is wrongly added to repair time.

Throughout we assume that all expectations are finite and all necessary regularity conditions hold.

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II. Unavailability of Gaver's Parallel System

Assuming that the two-unit system in Fig. 1 has independent exponential life lengths with the same failure rate (λ), the availability of Gaver's parallel system is given by the following equation:

$$\bar{u} = \frac{2 - L_R}{2\lambda E(R) + L_R} \tag{1}$$

or equivalently, the unavailability is found as:

$$u = 1 - \bar{u} = 2 - 2[\frac{\lambda ER + 1}{2\lambda ER + L_R}] \tag{2}$$

where terms are defined in the notation section above. The detailed derivation of this formula can be found in [1].

III. Unavailability with Travel Time

As mentioned, Gaver's formula does not take traveling time into account. Now, let us consider that the travel time is added to the first repair time only. (See Figure 2.) Then, the unavailability of the two-unit parallel system can be calculated as

$$u = 1 - \left[\frac{2\left[\frac{1}{L_R} - L_T\right] + 1}{1 + 2\lambda E(T+R) + 2\lambda ER\left[\frac{1}{L_R} - L_T\right]} \right] \tag{3}$$

We derive this formula as follows.

Consider a stochastic process {X(t), t≥0} with state space {1, 2, 3} corresponding to the states of the two-unit parallel system over time. Let state 3 be the "duplex" state where the two units are working, state 2 be the "simplex" state where only one unit is working and state 1 be the "down" state where both units are not working. Then since both units have an exponential distribution which has the memoryless property the process $\{X(t), t \geq 0\}$ is a regenerative process with regeneration time points $G_1, G_2, G_3, \ldots$, which correspond to points of return to the duplex state. (See [12] pages 425-428 for a discussion of regenerative processes.) The points $G_1, G_2, G_3, \ldots$, are such that continuation of the process beyond $G_1, G_2, G_3, \ldots$, will probabilistically replicate the whole process starting at time t = 0.

We can construct the process $\{X(t), t \geq 0\}$ as shown in Figure 2. We let U, T, S, S_1, $S_2, \ldots$, R, R_1, $R_2, \ldots$ be independent random variables such that (a) U has the exponential distribution with rate 2λ, (b) S, S_1, $S_2, \ldots$, have the exponential distribution with rate λ, (c) R, R_1, $R_2, \ldots$ have the repair distribution F_R and (d) T has the travel time distribution F_T. Then one can construct the process $\{X(t), t \geq 0\}$ as follows:

$$X(t) = \begin{cases} 3 & \text{if } t < U \text{ or } U + T + \sum_{i=1}^{N} R_i \leq t \\ 2 & \text{if } U \leq t < U + S_1 \\ & \text{or } U + T + R_1 \leq t < U + T + R_1 + S_2, \ldots, \\ & \text{or } U + T + \sum_{i=1}^{N-2} R_i \leq t < U + T + \sum_{i=1}^{N-2} R_i + S_{N-1}, \\ & \text{or } U + T + \sum_{i=1}^{N-1} R_i \leq t < U + T + \sum_{i=1}^{N} R_i, \\ 1 & \text{Otherwise} \end{cases}$$

See Figure 2 for a typical sample path of this process.

Then, as stated in [12] page 426, the availability of the two-unit parallel system is given by

$$\bar{u} = \frac{E\left[\text{amount of time in state } 2 \text{ and } 3 \text{ during one cycle}\right]}{E\left[\text{time of one cycle}\right]}$$

To calculate this expectations, let N=1, if $S_1 \geq T+R_1$; N=2, if $S_2 \geq R_2$, $S_1 < T+R_1$; and in general, N=n if $S_n \geq R_n$, $S_i < R_i$, i=2, 3, …, n-1, $S_1 < T+R_1$. Note N can be interpreted as the number of unit failures before the return to the duplex state. In Fig. 2, N=4.

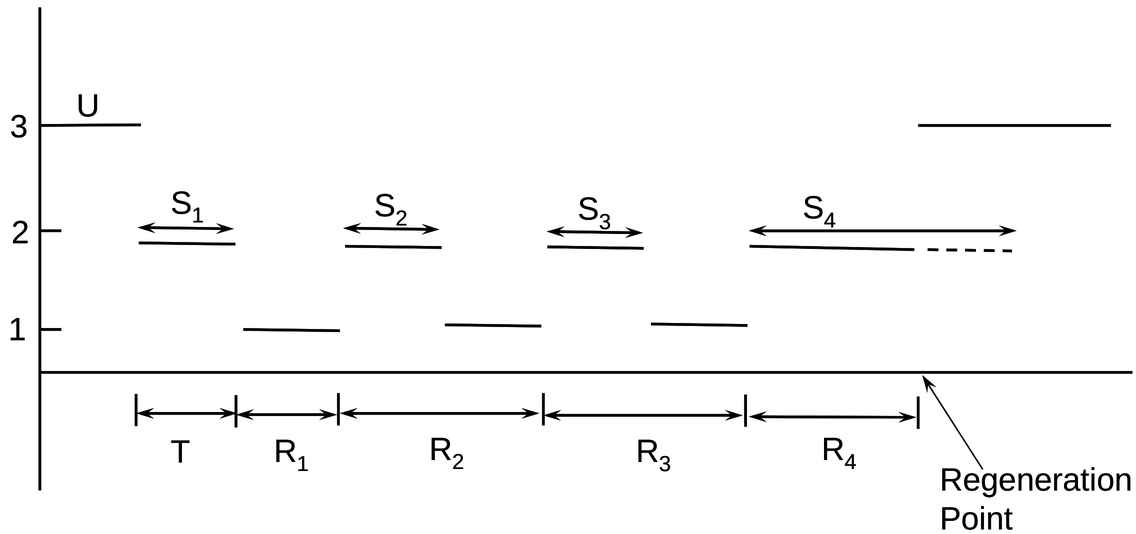

Figure 2. Typical sample path of the process X. Note that the "number of process failures before the return to the duplex state" is 4 in this sample path.

Before calculating the expectations in this equation, we also need a lemma. Throughout let $L_T^{'}$ be the derivative of the Laplace transform evaluated at λ, i.e.,

$$L_T^{'} = -\int_0^{\infty} te^{-\lambda t} dF_T(t)$$

Lemma: The following identities hold.

(a) $EN = \frac{1}{L_R} + (1 - L_T)$

(b) $E(TI[N=1]) = -L_R L_T^{'}$, I[N=1] is the indicator function of the event [N=1]

(c) $E[S-R|S>T+R]=\frac{1}{\lambda}-\frac{L_T^{'}}{L_T}$

(d) $E[S_N-R_N]=\frac{1}{\lambda}-L_R L_T^{'}$

Proof of (a): Note that

$$EN=\sum_{n=0}^{\infty}P[N>n]$$

and that

$$P[N>0]=1$$

$$\begin{aligned}P[N>1]&=P[S_1<T+R_1]\\&=P[S<T+R]\end{aligned}$$

$$\begin{aligned}P[N>2]&=P[S_1<T+R_1,\ S_2<R_2]\\&=P[S_1<T+R_1]P[\mathrm{S}_2<R_2]\\&=P[S<T+R]P[S<R]\end{aligned}$$

In general, for n > 2, we have

$$\begin{aligned}P[N>n]&=P[S_1<T+R_1,S_2<R_2,\ldots,S_n<R_n]\\&=P[S_1<T+R_1]P[S_2<R_2]\ldots P[S_n<R_n]\\&=P[S<T+R]P[S<R]^{n-1}\end{aligned}$$

Hence,

$$\begin{aligned}EN&=\sum_{n=0}^{\infty}P\left[N>n\right]\\&=1+P[S<T+R]\sum_{n=0}^{\infty}P[S<R]^n\\&=1+P[S<T+R]/(1-P[S<R])\\&=1+(1-P[S\geq T+R])/P[S\geq R]\\&=1+(1-L_{T+R})/L_R\\&=1+(1-L_T L_R)/L_R\\&=1+[1/L_R]-L_T\end{aligned}$$

since $L_X = P\left[S > X\right]$ for any positive random variable X. We also have used the formula for the summation of a geometric series. Identity (a) follows.

Proof of (b): We have

$$
\begin{aligned}
E(TI[N=1]) &= E(TI[S_1 > T + R_1]) \\
&= \int_0^\infty tP[S > t + R]dF_T(t) \\
&= \int_0^\infty tL_{t+R}dF_T(t) \\
&= \int_0^\infty te^{-\lambda t}L_R dF_T(t) \\
&= L_R\int_0^\infty te^{-\lambda t}dF_T(t) \\
&= -L_R L_T^{'}
\end{aligned}
$$

Identity (b) follows.

Proof of (c): We have

$$
\begin{aligned}
&E[S-R \mid S>T+R] \\
&\quad = E[S-R \mid S-R>T] \\
&\quad = \int_0^\infty P[S-R>u \mid S-R>T]\,du \\
&\quad = \frac{\int_0^\infty P[S-R>\max\{u,T\}]\,du}{P[S-R>T]} \\
&\quad = \frac{\int_0^\infty P[S>R+\max\{u,T\}]\,du}{P[S>R+T]} \\
&\quad = \frac{\int_0^\infty \left[L_{R+\max\{u,T\}}\right]du}{L_{R+T}} \\
&\quad = \frac{L_R\int_0^\infty \left[L_{\max\{u,T\}}\right]du}{L_R L_T} \\
&\quad = \frac{1}{L_T}\left\{\int_0^\infty\left[e^{-\lambda u}F_T(u)+\int_u^\infty e^{-\lambda t}dF_T(t)\right]du\right\} \\
&\quad = \frac{1}{L_T}\left\{\int_0^\infty e^{-\lambda u}F_T(u)du+\int_0^\infty\int_u^\infty e^{-\lambda t}dF_T(t)du\right\} \\
&\quad = \frac{1}{L_T}\left\{\frac{1}{\lambda}\left[\int_0^\infty e^{-\lambda u}dF_T(u)+F_T(0)\right]+\int_0^\infty\int_0^t du\ e^{-\lambda t}dF_T(t)\right\} \\
&\quad = \frac{1}{L_T}\left\{\frac{1}{\lambda}\left[\int_0^\infty e^{-\lambda t}dF_T(t)\right]+\int_0^\infty te^{-\lambda t}dF_T(t)\right\} \\
&\quad = \frac{1}{L_T}\left\{\frac{1}{\lambda}L_T+\int_0^\infty te^{-\lambda t}dF_T(t)\right\} \\
&\quad = \frac{1}{\lambda}-\frac{L_T^{'}}{L_T}
\end{aligned}
$$

were we have used integration by parts and changed the order of integration.

Identity (c) follows.

Proof of (d): We have

$$
\begin{aligned}
E[S_N - R_N] &= E[S_1 - R_1 | N = 1]P[N = 1] \\
&\quad + \sum_{n=2}^{\infty} E[S_n - R_n | N = n]P[N = n] \\
&= E[S_1 - R_1 | S_1 > T + R_1]P[N = 1] \\
&\quad + \sum_{n=2}^{\infty} E[S_n - R_n | S_1 < T + R_1, S_2 < R_2, \dots, S_{n-1} < R_{n-1}, S_n > R_n]P[N = n] \\
&= E[S - R | S > T + R]P[N = 1] \\
&\quad + \sum_{n=2}^{\infty} E[S_n - R_n | S_n > R_n]P[N = n] \\
&= E[S - R | S > T + R]P[N = 1] \\
&\quad + E[S - R | S > R]\left[\sum_{n=2}^{\infty} P[N = n]\right] \\
&= (E[S - R | S > T + R] - E[S - R | S > R])\,P[N = 1] + E[S - R | S > R] \\
&= \left[\frac{1}{\lambda} - \frac{L_T'}{L_T} - \frac{1}{\lambda}\right] P[S_1 > T + R] + \frac{1}{\lambda} \\
&= \left[-\frac{L_T'}{L_T}\right] L_{T+R} + \frac{1}{\lambda} \\
&= -\frac{L_T'}{L_T} L_T L_R + \frac{1}{\lambda} \\
&= -L_R L_T' + \frac{1}{\lambda}
\end{aligned}
$$

Identity (d) follows and we finished proving the lemma.

Before we finish deriving the unavailability formula we need Wald's Equation given in [12] page 651.

Wald's equation: If $X, X_1, X_2, \ldots$ are independent and identically distributed random variables having finite expectation, and if M is a stopping time for $X_1, X_2, \ldots$, that is, if the event {M=m} is independent of $X_{m+1}, X_{m+2}, \ldots$, then if $E[M] < \infty$ we have

$$E\left[\sum_{1}^{M} X_m\right] = E[M]E[X]$$

Returning to the derivation of the unavailability formula we use Wald's Equation assuming that the appropriate expectation are finite to get

$$E\sum_{i=1}^{N} S_i = ENES \text{ and } E\sum_{i=1}^{N} R_i = ENER$$

Now, we are ready to calculate the unavailability formula using the construction of the process {X(t), t≥0} We have:

$$\begin{aligned}
\bar{u} &= \frac{E\,[amount\ of\ time\ in\ state\ 2\ and\ 3\ during\ one\ cycle]}{E\,[time\ of\ the\ cycle]} \\
&= \frac{E\left[U + \sum_{i=1}^{N-1} S_i + R_N + TI[N=1]\right]}{E\left[U + T + \sum_{i=1}^{N} R_i\right]} \\
&= \frac{EU + E\sum_{i=1}^{N} S_i - E[S_N - R_N] + ETI[N=1]}{EU + ET + E\sum_{i=1}^{N} R_i} \\
&= \frac{EU + ENES - E[S_N - R_N] + ETI[N=1]}{EU + ET + ENER}
\end{aligned}$$

And, since $u = 1 - \bar{u}$,

$$u = \frac{ET + EN(ER - ES) + E[S_N - R_N] - ETI[N = 1]}{EU + ET + ENER}$$

Hence, using the lemma we just proved, we have

$$u = \frac{ET + \left[\frac{1}{L_R} + (1 - L_T)\right]\left[ER - \frac{1}{\lambda}\right] + \left[-L_R L_T' + \frac{1}{\lambda}\right] - \left[-L_R L_T'\right]}{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + [1 - L_T]\right]ER}$$

$$= \frac{ET + \left[\frac{1}{L_R} + (1 - L_T)\right]\left[ER - \frac{1}{\lambda}\right] + \frac{1}{\lambda}}{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + [1 - L_T]\right]ER}$$

$$= \frac{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + (1 - L_T)\right]ER - \frac{1}{\lambda}\left[\frac{1}{L_R} + (1 - L_T)\right] + \frac{1}{2\lambda}}{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + [1 - L_T]\right]ER}$$

$$= \frac{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + (1 - L_T)\right]ER - \frac{1}{\lambda}\left[\frac{1}{L_R} + (1 - L_T)\right] + \frac{1}{2\lambda}}{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + [1 - L_T]\right]ER}$$

$$= 1 - \left[\frac{\frac{1}{\lambda}\left[\frac{1}{L_R} + (1 - L_T)\right] - \frac{1}{2\lambda}}{\frac{1}{2\lambda} + ET + \left[\frac{1}{L_R} + [1 - L_T]\right]ER}\right]$$

$$= 1 - \left[\frac{2\left[\frac{1}{L_R} + (1 - L_T)\right] - 1}{1 + 2(\lambda ET) + 2(\lambda ER)\left[\frac{1}{L_R} + [1 - L_T]\right]}\right]$$

$$= 1 - \left[\frac{2\left[\frac{1}{L_R} - L_T\right] + 1}{1 + 2\lambda E(T + R) + 2\lambda ER\left[\frac{1}{L_R} - L_T\right]}\right]$$

This finishes the derivation of the unavailability formula of the parallel system with a single traveling repairperson.

IV. Effect of Adding Travel Time to Repair Time

Let u be the exact unavailability as given by formula (3). Let u_R be the unavailability calculated under the assumption that the travel time is added to every repair time, as shown in Figure 3. The purpose of this section is to obtain bounds on the percent error that occurs when u_R is used to approximate u.

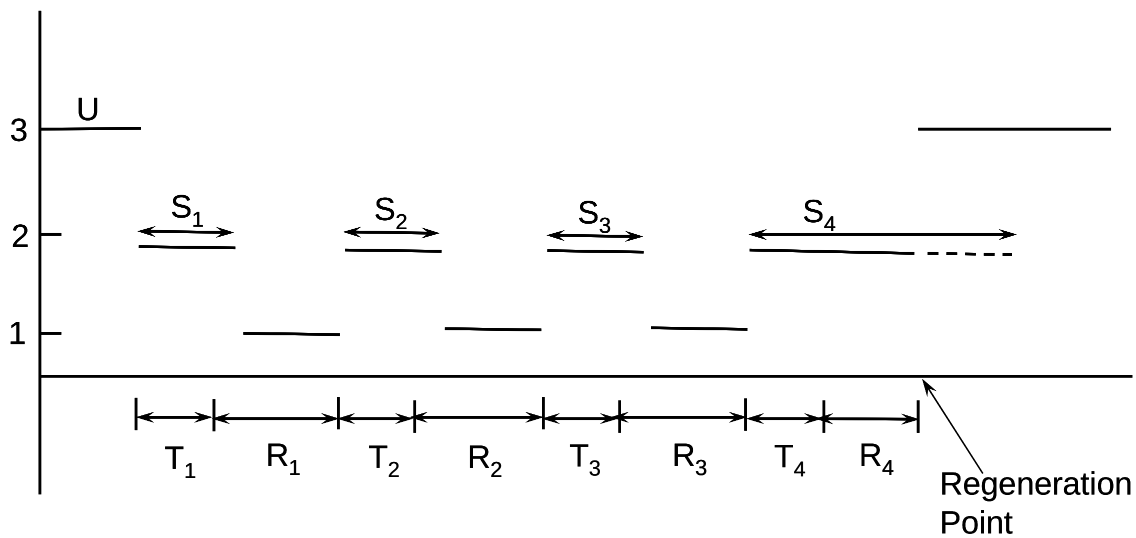

Figure 3. The sample path under the assumption that the travel time is part of every repair time.

Since the percent error is equal to

$$\frac{u_R - u}{u} \times 100 = \left[\frac{1}{(u/u_R)} - 1 \right] \times 100$$

it is enough to obtain bounds for u/u_R.

From formula (3) we have that

$$u_R = 1 - \left[\frac{2\left[\frac{1}{L_{T+R}} - 1\right] + 1}{1 + 2\lambda E(T+R) + 2\lambda E(T+R)\left[\frac{1}{L_{T+R}} - 1\right]} \right]$$

which can be simplified to

$$u_R = \frac{2\left[\lambda E(T+R) + L_{T+R} - 1\right]}{2\lambda E(T+R) + L_{T+R}} \tag{4}$$

In contrast, if we assume that the repair time is added to the travel time, that is, the first failure is corrected in a time with distribution F_{T+R} and the rest are corrected instantaneously as shown in Figure 4, then, from formula (3), we have that

$$u_T = 1 - \left[\frac{2\left[1 - L_{T+R}\right] + 1}{1 + 2\lambda E(T+R)} \right]$$

were u_T is the unavailability under the assumption above.

We can simplify the formula for u_T to

$$u_T = \frac{2\left[\lambda E(T+R) + L_{T+R} - 1\right]}{1 + 2\lambda E(T+R)} \tag{5}$$

According to the assumptions which lead to u_R and u_T, it follows that

$$u_T \le u \le u_R$$

We also have from formulas (4) and (5),

$$\frac{2\lambda E(T+R) + L_{T+R}}{2\lambda E(T+R) + 1} = \frac{u_T}{u_R} \le \frac{u}{u_R} \le 1 \tag{6}$$

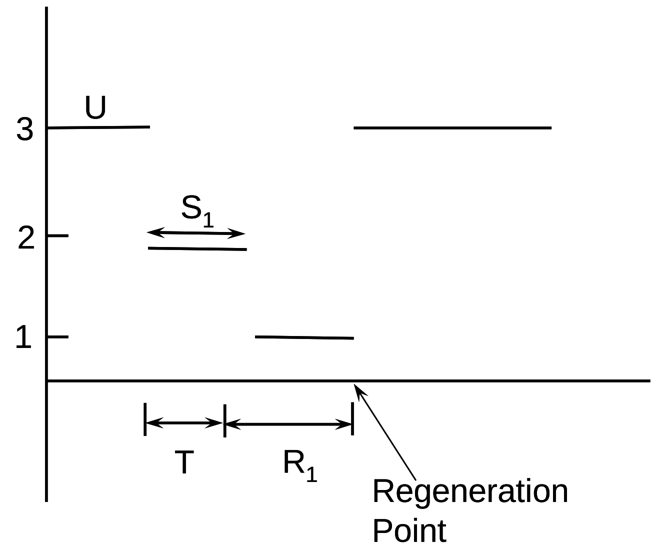

Figure 4. The sample path under the assumption that the repair time is added to the travel time, that is, the first failure is repaired in time with distribution F_{T+R} and the following failure are repaired in zero time.

The Laplace transform of a distribution increases as the tails of the distribution increase in weight (The appendix makes this statement precise and provides a proof). Since any distribution with mean E(T+R) has heavier tails than the distribution which degenerates at E(T+R), it follows that the smallest value of the quantity

$$\frac{2\lambda E(T+R)+L_{T+R}}{2\lambda E(T+R)+1}$$

occurs when the total travel and repair time is constant. Since in this case

$$L_{T+R}=e^{-\lambda E(T+R)}$$

we have the following result

$$\frac{2\lambda E(T+R)+e^{-\lambda E(T+R)}}{2\lambda E(T+R)+1}\leq\frac{u}{u_R}\leq 1 \qquad (7)$$

Let $x = \lambda E(T+R)$ and $f(x) = \dfrac{2x + e^{-x}}{2x+1} \quad (x \geq 0)$.

Let m be the solution to the equation $\dfrac{df(x)}{dx} = 0$ and we have

$$m = \ln(m + \frac{3}{2})$$

that is, m=0.8577. It is easy to show that the function f is decreasing from 0 to m and increasing afterwards. It follows that the expression on the left-hand side of the inequality (7) achieves its minimum when

$$\lambda E(T+R) = m = 0.8577$$

And since $f(m)$=0.7879, we have the following inequality. From (7) we have

$$0.7879 \leq \frac{u}{u_R} \leq 1 \text{ which implies that}$$

$$0 \leq \frac{u_R - u}{u} \leq 0.2692 \qquad (8)$$

The percent overestimation of u by u_R as a function of $x = \lambda E(T+R)$ can be expressed as

$$e = \frac{u_R - u}{u} \times 100 = \frac{1 - e^{-x}}{2x + e^{-x}} \times 100, \quad where \ \ x = \lambda E(T+R) \qquad (9)$$

From inequality (8), it is seen that the maximum overestimation is 0.2692. Figure 5 plots the percent overestimation as a function of log(x). It can be observed that the maximum possible percent overestimation 26.9% occurs when $x = \lambda E(T+R)$=0.86

From the expression (9) the maximum possible percent overestimation of u by u_R can be evaluated for different ranges of $\lambda E(R+T)$:

1. $\lambda E(T+R) \le 10^{-2}$ implies $e \le 0.99\%$.

2. $\lambda E(T+R) \le 10^{-3}$ implies $e \le 0.10\%$.

3. $\lambda E(T+R) \le 10^{-4}$ implies $e \le 0.01\%$.

4. $\lambda E(T+R) \le 10^{-5}$ implies $e \le 0.001\%$.

Notice that $\lambda E(T+R)$ is equal to $\dfrac{E(T+R)}{1/\lambda} = \dfrac{E(T+R)}{\text{Mean unit life}}$ which is usually small in practice. From the observations 1 to 4, it is seen that, in practice, not very much harm is done by adding travel time to repair time.

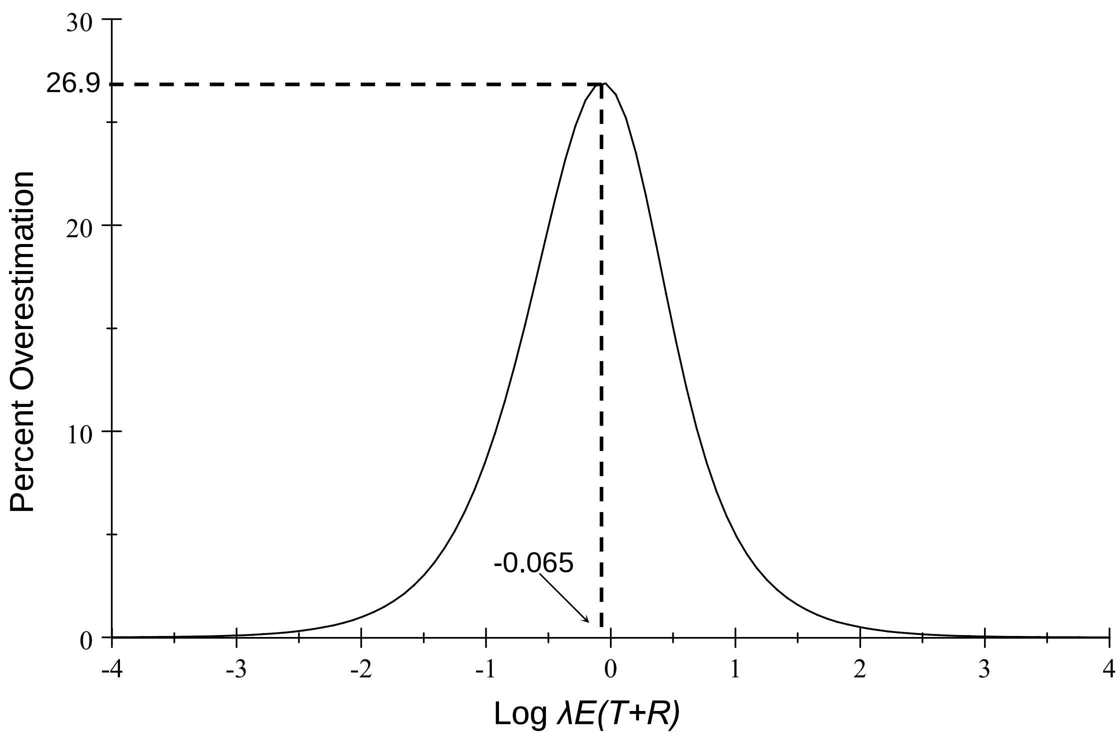

Figure 5. Maximum possible percent overestimate of u by u_R expressed as a function of the product of the units failure rate λ and the mean total travel and repair time E(T+R). The absolute maximum of 26.9% occurs when λE(T+R) = 0.86, i.e., when log(λE(T+R) = -0.065.

V. Conclusions

The unavailability of Gaver's two-unit parallel system is revisited. His formula for the unavailability is

$$u = 1 - \bar{u} = 2 - 2[\frac{\lambda ER + 1}{2\lambda ER + L_R}]$$

The unavailability formula for the two-unit parallel system with the traveling time taken into consideration is derived to be:

$$u = 1 - \left[\frac{2\left[\frac{1}{L_R} - L_T\right] + 1}{1 + 2\lambda E(T+R) + 2\lambda ER\left[\frac{1}{L_R} - L_T\right]} \right]$$

Then, the effect of combining traveling time with repair time is calculated and discussed. In particular we have

$$0 \le \frac{u_R - u}{u} \le 0.2692 .$$

where u_R is the unavailability calculated by adding travel time to repair time.

That is, the maximum percent overestimation of u by u_R does not exceed 27%. Further we show that if $\lambda E(T+R)$ is small, as it usually is in practice, the percent overestimation is also small. (See results 1 to 4 in Section IV for a precise version of this statement.) It follows that in practice not very much harm is done by combining travel time and repair time.

Appendix

Let F and G be the distribution of two non-negative random variables with the same mean. Then

$$\int_0^\infty \overline{F}(t)dt = \int_0^\infty \overline{G}(t)dt,$$

where $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. See for example [13]. Since the nonincreasing function $\overline{F}$ and $\overline{G}$ enclose the same area below in the first quadrant, it follows that $\overline{F}$ must cross $\overline{G}$ at least once. We say that G has heavier tails than F if $\overline{F}$ crosses $\overline{G}$ exactly once from above (see Figure 6).

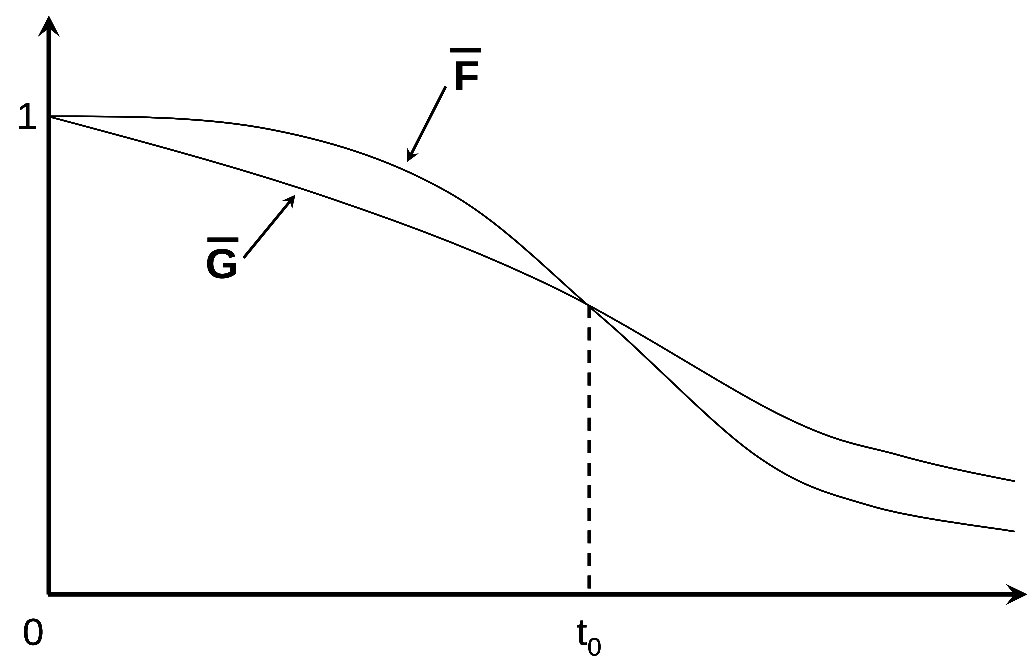

Figure 6. The distribution G has heavier tails than the distribution F. Note that the area below $\overline{G}$ and $\overline{F}$ in the first quadrant is the same for both. The point where $\overline{F}$ and $\overline{G}$ cross is denoted by t_0.

Theorem: The Laplace transform of a distribution with a given mean increases as the tails of the distribution increase in weight. That is, if the distribution G has a heavier tail than the distribution F, then

$$L_F \leq L_G$$

L_F and L_G are the Laplace transforms of the distribution F and G.

Proof: Let $L_F(\lambda)$ be the Laplace transform of the distribution F evaluated at λ, i.e.,

$L_F(\lambda) = \int_0^\infty e^{-\lambda t} dF(t)$. Let $M_F(\lambda)$ be the transform of F defined by,

$$M_F(\lambda) = \int_0^\infty e^{-\lambda t} \overline{F}(t) dt .$$

Using integration by parts we can show that

$$L_F(\lambda) = 1 - \lambda M_F(\lambda) .$$

Similarly, we have

$$L_G(\lambda) = 1 - \lambda M_G(\lambda) .$$

It follows that, to show $L_F(\lambda) \leq L_G(\lambda)$, it is only necessary to show

$$M_F(\lambda) \geq M_G(\lambda) .$$

Let t_0 be the point where $\overline{F}$ crosses $\overline{G}$ (see Figure 6). Then we have since

$$\int_0^\infty \overline{F}(t) dt - \int_0^\infty \overline{G}(t) dt = 0,$$

that

$$M_F(\lambda) - M_G(\lambda) = \left[\int_0^\infty e^{-\lambda t}\,\overline{F}(t)dt - \int_0^\infty e^{-\lambda t}\,\overline{G}(t)dt \right]$$
$$- e^{-\lambda t_0}\left[\int_0^\infty \overline{F}(t)dt - \int_0^\infty \overline{G}(t)dt \right]$$

Equivalently,

$$M_F(\lambda) - M_G(\lambda) = \int_0^\infty \left[e^{-\lambda t} - e^{-\lambda t_0}\right]\left[\overline{F}(t) - \overline{G}(t)\right]dt\,.$$

But the functions $\left[e^{-\lambda t} - e^{-\lambda t_0}\right]$ and $\left[\overline{F}(t) - \overline{G}(t)\right]$ always have the same sign, since $e^{-\lambda t}$ is decreasing, and t_0 is the crossing point of $\overline{F}$ and $\overline{G}$. It follows that

$$M_F(\lambda) - M_G(\lambda) \geq 0\,.$$

Therefore the conclusion of the theorem follows.

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Authors

Ramón V. León received his Ph.D in 1980 from the Florida State University. From 1991, he has been an associate professor at the University of Tennessee, Knoxville. Prior to that from 1980 to 1991, he was a Member of Technical Staff and then supervisor at Bell Laboratories where he had responsibility for research and consulting in reliability data analysis and modeling. He was chair of the 2006 Quality and Productivity Research Conference. He has published in journals like *Technometrics*, Annals of Probability, *Journal of Quality Technology* and others. He is the associate editor of the American Society for Quality (ASQ) magazine "Six Sigma Forum Magazine."

Frank M. Guess is Full Professor of Statistics in the College of Business Administration at the University of Tennessee, where he served as the CBA Statistics PhD Director from 1999 to 2004. He earned the Frank Wilcoxon's Best First Year Student in Applied Statistics Award in 1979 and the Ralph Bradley PhD Student Award for outstanding achievement in 1984, both at Florida State University's Department of Statistics. Dr. Guess has had grants from IBM, Air Force Office of Scientific Research, and the Army Office of Research. He has been an Invited Program Committee Member at the Fourth International Reliability Research Conference held at the University of Missouri, Columbia, June, 1991 and also served as an invited Program Committee Member of the First International Research Conference on Lifetime Data Models held at Harvard University, co-sponsored by the Statistics Department at Harvard University, June, 1994. More recently, he served on the Program Committee for the International Workshop on Reliability and Its Applications in Seoul, Korea, December 2003, co-sponsored by the

United States National Science Foundation and the Korean Reliability Society (KORAS) in cooperation with Rutgers University, USA and Hanyang University, Korea. He was on the Invited Program Committee in 2005 for the 10th Anniversary MIT's Information Quality Conference, plus also serves on the Program Committee for 2006 and 2007 MIT conference. He is on the Editorial Board of the *International Journal of Reliability and Applications.* He has written over 30 papers in reliability and statistics in *Annals of Statistics*, *Journal of Statistical Planning & Inference, Quality and Reliability Engineering: International, IE News: Quality Control.*

Kaixiang Tao is currently a graduate teaching assistant in the Department of Statistics, Operations and Management Science at University of Tennessee, Knoxville. He received his Ph.D in Materials Science and Engineering in 2006 and has published about 10 articles in the field of materials research with the support of the IMI (International Materials Institute) program of National Science Foundation.