QUANTUM COMPUTATION OF SCATTERING AMPLITUDES

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Nature is quantum.

- hard to understand for us classical beings.
- no complete description of what quantum means.
  - need (classical) concept of measurement to make sense of quantum.

By understanding quantum, we hope to be able to take advantage of everything Nature has to offer.

▶ One such advantage seems to be in IT.
Information and entropy

What does information have to do with Physics?

- Computers process information. They need power to operate. They get hot. Why?
- Thermodynamics: processes in Nature are irreversible. Entropy always increases, heat flows from hot to cold, etc.
- But laws of Nature are invariant under time reversal.
  - At a fundamental level, all processes are reversible ($\Delta S = 0$).

Is processing of information an irreversible (dissipative) process?

ANSWER: No!

EXAMPLE: The NAND gate, $(a, b) \mapsto \overline{a \land b}$

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▶ Irreversible!

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Add a bit and do *Toffoli gate*, instead..

\[ T : (a, b, c) \mapsto (a, b, c \oplus (a \land b)) \]

- flips \( a \land b \), if \( c = 1 \), \( \therefore \) NAND gate.

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Reversible! \( T^2 = \mathbb{I} \), so \( T^{-1} = T \)

Then what costs energy (entropy)?

**ANSWER:** The *erasure* of information.
Landauer’s principle (1961)
Erasing 1 bit of information requires entropy $\Delta S = k_B \log 2$.
- Minimum cost in operating a computer.
- Present computers far from this limit ($\Delta S \sim 500k_B \log 2$).
- As computers get smaller, this limit will become significant.

Do we have to erase information?
ANSWER: No!

- At the end of a reversible computation, computer can reverse all steps and return to its initial state.
  - No junk to dispose of!
  - No energy loss!
  - No entropy generated!

With computers near Landauer limit, all information processing will have to be done reversibly, otherwise wires will melt.
Quantum information

You can extract information from a classical system without disturbing it.

- Can’t do that with a quantum system.

- FOOD FOR THOUGHT: Aren’t all systems quantum?

You measure observable $A$ and system collapses to an eigenstate of $A$.

- Worse: if $[A, B] \neq 0$, then measurement of $A$ will influence subsequent measurement of $B$.

Outcome of measurement is random:

- Let $A|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$.
  If system is in state $|\psi\rangle$, then outcome of measurement of $A$ is $\lambda_i$ with probability
  
  $$P_i = |\langle \lambda_i | \psi \rangle|^2$$

  - State $|\psi\rangle$ collapses to $|\lambda_i\rangle$.
  - We have no way of figuring out $|\psi\rangle$. 

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Qubits

Classical information comes in *bits*, which take values 0 or 1.

Quantum information comes in *qubits*, i.e.,

\[ |\psi\rangle = a|0\rangle + b|1\rangle \]

Probabilistic outcome: A measurement will project onto \( |0\rangle \) or \( |1\rangle \) with probability, respectively,

\[ P_0 = |a|^2, \quad P_1 = |b|^2 \]

EXAMPLES: spin-1/2 particle (\( |0\rangle = |\uparrow\rangle, |1\rangle = |\downarrow\rangle \)).

photon polarization (\( |0\rangle = |L\rangle, |1\rangle = |R\rangle \)).

With \( N \) qubits, state is superposition of \( \{ |x\rangle, x = 0, 1, \ldots, 2^N - 1 \} \)

- \( x \) in binary notation, e.g., for \( N = 3 \),

\[ x = 000, 001, 010, 011, 100, 101, 110, 111 \]
Quantum computation

1. Prepare initial state of $N$ qubits.
2. Evolve state by applying a string of quantum gates
   - evolution operators $U$ ($2^N \times 2^N$ unitary matrices, $U^\dagger U = I$)
3. Perform a measurement on the final state.

Efficiency: with $N = 100$ qubits, we can naturally (through quantum evolution) implement $10^{30} \times 10^{30}$ matrices ($2^{100} \sim 10^{30}$).

- Try that on a classical computer!
- Hilbert spaces are enormous!
Errors
A quantum algorithm is useful and goes beyond classical, if it can take advantage of *nonlocal correlations* (entanglement).

- $|01\rangle$ and $|10\rangle$ are not entangled.
- $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is entangled.
  - Einstein-Podolsky-Rosen (EPR) paradox.
- Extremely fragile correlations.
  - decay *very* rapidly, due to interactions with environment.
  - some (most) information lies in correlations with environment and cannot be accessed.

Schrödinger’s cat

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}} (|\text{dead}\rangle + |\text{alive}\rangle)$$

Didn’t like it: all cats he had observed were either $|\text{dead}\rangle$ or $|\text{alive}\rangle$.

Why?
Cat interacts with environment and information is *immediately* transfered to correlations with environment.

- Environment *measures* cat continuously, projecting it onto states we are familiar with (*decoherence*).

For QC, we need a $|\text{cat}\rangle$-like state, except not as large.

- Need to deal with errors due to decoherence.
  - the state will generically interact with environment and decay very rapidly
  - computer will immediately crash.
Shor’s algorithm (1994)

A milestone.

- Showed that QC is by far superior to a classical computer, because it can find prime factors of a given number very efficiently.

*Intractable problem:* hard to find solution, but easy to verify once found.

- Let $N = pq$, $p \sim 2^m$, $q \sim 2^n$.
- Need $\sim mn \sim \log p \log q \sim (\log N)^2$ steps (time) to verify $pq = N$.
- Given $N$, to find $p$ and $q$, best algorithm (*number field sieve*) takes superpolynomial time

$$t \approx C \exp \left\{ \left[ \frac{64}{9} \log N (\log \log N)^2 \right]^{1/3} \right\}$$

Experimentally, for $N \sim 10^{130}$, using a few hundred workstations, $t \approx 1$ month, so $C \approx 1.5 \times 10^{-18}$ months.

For a 400-digit number, we need

$$t \approx 2.6 \times 10^{11} \text{ months} \sim 10^{10} \text{ years} \sim \text{age of the Universe!}$$

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Shor’s quantum algorithm takes *polynomial* time to run,

\[ t \approx C'(\log N)^3 \]

If a (future!) QC matches the performance of a classical computer for a 130-digit number, then \( C' \approx 4 \times 10^{-8} \) months, and for a 400-digit number, we need

\[ t \approx 29 \text{ months} \sim 2.5 \text{ years} \]

Huge (exponential) improvement!
QFT
NEWTONIAN SPACE AND TIME

\[ \delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 \]

invariant under rotations

inertial observers feel no forces.

\[ \vec{F} = m\vec{a} \]

invariant under

\[ \vec{x} \rightarrow \vec{x} - \vec{v}t \]

(transformation law between inertial observers - Galilean).

★ Newton is inertial; apple isn’t \((\vec{a} = -g)\).
ELECTROMAGNETISM

Unification of electricity and magnetism (Maxwell)

⇒ waves

\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \]

they travel at speed of light

\[ c = 3 \times 10^8 \text{m/s} \]

regardless of frame of reference

∴ incompatible with Newton’s laws.

Lorentz transformation (boost in x-direction):

\[ x \rightarrow \gamma(x - vt) \quad t \rightarrow \gamma(t - vx/c^2) \]

(\(\gamma < 1\) - contraction) known before Einstein.

★ Radiation consists of fields.
**SPACETIME**

Einstein: space and time mix, form *space-time*

Invariant distance:

\[ \delta s^2 = -c^2 \delta t^2 + \delta x^2 + \delta y^2 + \delta z^2 = -c^2 \delta \tau^2 \]

\( \tau \): proper time.

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**SPECIAL RELATIVITY**

Far-reaching consequences:

\[ E = mc^2 \]
Blackbody spectrum:

\[ I(\nu) = \frac{8\pi kT\nu^2}{c^3} \leftarrow \text{WRONG!} \]

Planck in an act of despair proposes (light emitted as quanta)

\[ E = h\nu \]

changes an integral to a sum:

\[ I(\nu) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \leftarrow \text{CORRECT!} \]
Einstein explains photoelectric effect (1921 Nobel Prize)

- light is absorbed as quanta.

QUESTION: does light travel as particles (photons)?

- hard to swallow: photons can’t be hard balls - they interfere!

Einstein realized the consequences (unpredictability) before Heisenberg’s Uncertainty Principle

\[ \Delta p \Delta x \geq \hbar \]

- didn’t like it

“God does not play dice with the Universe”
QUANTUM MECHANICS + RELATIVITY

Klein-Gordon equation (discovered by Schrödinger)

\[ \nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = m^2 \Psi \]

- doesn’t work (negative probabilities)

Dirac equation

\[ i \gamma^\mu \partial_\mu \Psi = m \Psi \]

- leads to hole theory nonsense.

Finally, Quantum Field Theory - a triumph!

\( \Psi \) is not a wavefunction (probability amplitude), but a field, similar to electric and magnetic fields.

- Explains Pauli exclusion principle
- Predicts anti-matter
GRAVITY + RELATIVITY

Einstein: The apple is the inertial observer, not Newton!
- Newton feels a force, apple doesn’t.

Principle of Equivalence

Apple travels along geodesic

\[ x \approx -\frac{1}{2}gt^2 \]

\[ ds^2 = -c^2d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \]

mass (energy) creates spacetime
- time “warp” factor:

\[ g_{00} \approx 1 - \frac{2gR}{c^2} = 1 - 1.4 \times 10^{-9} = 0.9999999986 \]
GENERAL RELATIVITY

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

Instant success:

★ Mercury precession
observed: \( \Delta \phi = 5,601'' / \text{century} \).
geocentric system effects: 5,026''/century.
other planets: 532''/century.
Einstein: 43''/century.

★★ bending of light by Sun

\[ \Delta \theta \simeq \frac{4GM_\odot}{c^2 R_\odot} \simeq 1.75'' \]
observed by Eddington (1919)

★★ Big-bang nucleosynthesis (BBN) (Universe \( \sim 1 \) sec old)
QUANTUM MECHANICS + ELECTROMAGNETISM

coupling (fine-structure) constant

\[ \alpha = \frac{e^2}{2hc} = \frac{1}{137} \]

QED - infinities
vacuum polarization modifies Coulomb Law

\[ V(r) = \frac{e(r)}{4\pi r} , \quad e(r) = e \left\{ 1 + \frac{2\alpha}{3\pi} \ln \frac{\lambda_e}{r} + \ldots \right\} \]

for \( r \ll \lambda_e = h/m_e c \) (Compton wavelength).

★ running coupling constant (vacuum is a dielectric)
OTHER FORCES

WEAK: $\beta$ decay

$$n \rightarrow p + e^- + \bar{\nu}_e$$

point interaction with coupling (Fermi)

$$G_F = 1.166 \times 10^{-5} (\hbar c)^3 / GeV^2$$

needs modification at high energies (infinite probabilities)

Introduce weak ($W$) boson (massive) mediating interaction.

$$\frac{G_F}{(\hbar c)^3} = \frac{\sqrt{2} g^2}{8 m_W^2 c^4}, \quad m_W = 80.425 \ GeV/c^2$$

weak “fine-structure” constant (dimensionless):

$$\alpha_W = \frac{g^2 \hbar^3}{4 \pi c} = \frac{1}{29}$$

$$\alpha / \alpha_W = \sin^2 \theta_W \ (\text{Weinberg angle}).$$
STRONG: Quantum Chromodynamics (QCD)

Quarks and gluons have colors
▶ cannot be free (directly detected)

Only color neutral objects can be “seen”
(nucleons, mesons, etc)

Scale:

\[ \Lambda_{QCD} = 0.236 \text{ GeV} \]

★ asymptotic freedom

At high energies: coupling constant decreases
▶ spin-1 charged bosons (gluons) make vacuum paramagnetic

[Politzer; Gross and Wilczek - Nobel Prize 2004]

E&M + WEAK + STRONG = STANDARD MODEL
QUANTUM MECHANICS + GRAVITY

Newton’s constant $G$ is like weak (Fermi) constant $G_F$. It needs to be expressed in terms of a massive boson

$$G \sim \frac{g^2}{m_P^2}, \quad m_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19} \text{ GeV}/c^2$$

$P$ for Planck.

$m_P$ is the scale where quantum effects are expected to become important.
Scattering

Scattering process

\[ |1, \vec{k}_1; \cdots; M, \vec{k}_M \rangle_{t \rightarrow -\infty} \rightarrow |1', \vec{k}_1'; \cdots; N', \vec{k}_N' \rangle_{t \rightarrow +\infty} \]

Scattering amplitude

\[ \mathcal{A} = _{t \rightarrow +\infty} \langle M, \vec{k}_M; \cdots; 1, \vec{k}_1 | 1', \vec{k}_1'; \cdots; N', \vec{k}_N' \rangle_{t \rightarrow +\infty} \]

To calculate cross section \(|\mathcal{A}|^2\), apply quantum algorithm:


1. Prepare initial state \(|1, \vec{k}_1; \cdots; M, \vec{k}_M \rangle_{t \rightarrow -\infty}\) (eigenstate of free theory).
2. Adiabatically evolve it to an interacting theory eigenstate.
3. Evolve it for the duration of scattering with (unitary) quantum gates.
4. Adiabatically evolve it back to free theory state.
5. Measure the number of momentum modes (detector simulation).

Exponential improvement over lattice field theory on classical computers!
Harmonic oscillator

Set $\hbar = c = 1$.

Operators $q$ (position) and $p$ (momentum) satisfy commutation relations

$$[q, p] = i$$

The Hamiltonian is (setting the mass $m = 1$, for simplicity)

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2$$

where I added a constant to shift the ground state energy to zero.

Next, introduce

$$a \equiv \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p$$

We have

$$[a, a^\dagger] = 1$$

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \quad p = -i\frac{\sqrt{\omega}}{2}(a - a^\dagger)$$
Therefore, with *normal ordering*

\[ H = \omega a^\dagger a \]

Eigenvalue problem solved

\[ H |n\rangle = E_n |n\rangle , \quad E_n = n\omega \]

Can be simulated with \( N \) qubits, if mapped onto basis states

\(|n\rangle, n = 0, 1, \ldots, 2^N - 1.\)
Alternative simulation

Working in the \( q \) representation, we have

\[
p = -i \frac{d}{dq}, \quad a = \sqrt{\frac{\omega}{2}} q + \frac{1}{\sqrt{2\omega}} \frac{d}{dq}
\]

The ground state obeys

\[
a \Psi_0(q) = \sqrt{\frac{\omega}{2}} q \Psi_0(q) + \frac{1}{\sqrt{2\omega}} \psi'_0(q) = 0
\]

\[
\therefore \quad \Psi_0(q) = C e^{-\omega q^2 / 2}.
\]

This Gaussian can be simulated by \( N \) qubits.

Other states can then be generated.

For 1st excited state, introduce ancillary qubit and define

\[
H_1 = a^\dagger |1\rangle \langle 0| + a |0\rangle \langle 1|
\]

We have

\[
H_1 \Psi_0 |0\rangle = \Psi_1 |1\rangle, \quad H_1 \Psi_1 |1\rangle = \Psi_0 |0\rangle
\]

therefore

\[
e^{-i H_1 \pi / 2} \Psi_0 |0\rangle = -i \Psi_1 |1\rangle
\]
Free scalar field theory

Now consider a free scalar field $\phi(\vec{x})$, where $\vec{x} \in \Omega$, which is a cubic lattice in $d$ spatial dimensions with length $L$ and lattice spacing $a$

$$\Omega = a \mathbb{Z}^d_L/a$$

Let $\vec{a}_i (i = 1, \ldots, d)$ denote the basis of the lattice $\Omega (|\vec{a}_i| = a)$.

To compare with Nature, presumably we need to take the continuum limit

$$a \rightarrow 0$$

Let $\pi(\vec{x})$ be the conjugate field, satisfying commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = \frac{i}{ad} \delta_{\vec{x},\vec{y}}$$

The discretized gradient is defined by

$$\nabla_i \phi(\vec{x}) = \frac{\phi(\vec{x} + \vec{a}_i) - \phi(\vec{x})}{a} \quad (i = 1, \ldots, d)$$
The discretized Laplacian is
\[ \nabla^2 \phi(\vec{x}) = \sum_{i=1}^{d} \nabla_i^2 \phi(\vec{x}) = \sum_{i=1}^{d} \frac{\phi(\vec{x} + \vec{a}_i) + \phi(\vec{x} - \vec{a}_i) - 2\phi(\vec{x})}{a^2} \]

The Hamiltonian is
\[ H_0 = \frac{a^d}{2} \sum_{\vec{x} \in \Omega} \left( \pi^2(\vec{x}) + \phi(\vec{x})(-\nabla^2 + m^2)\phi(\vec{x}) \right) \]

Next, introduce the dual lattice \( \Gamma = \frac{2\pi \mathbb{Z}^d}{L/a} \) and the annihilation operator
\[ a(\vec{k}) = a^d \sum_{\vec{x} \in \Omega} e^{-i\vec{k} \cdot \vec{x}} \left[ \sqrt{\frac{\omega(\vec{k})}{2}} \phi(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}} \pi(\vec{x}) \right] \]

where \( \vec{k} \in \Gamma \), and
\[ \omega^2(\vec{k}) = e^{-i\vec{k} \cdot \vec{x}}(-\nabla^2 + m^2)e^{i\vec{k} \cdot \vec{x}} \]
\[ = m^2 + \frac{4}{a^2} \sum_{i=1}^{d} \sin^2 \frac{k_i a}{2} \]
We have

\[ [a(\vec{k}), a^{\dagger}(\vec{k}')] = L^d \delta_{\vec{k}, \vec{k}'} \]

\[
\phi(\vec{x}) = \frac{1}{L^d} \sum_{\vec{k} \in \Gamma} e^{i\vec{k} \cdot \vec{x}} \frac{1}{\sqrt{2\omega(\vec{k})}} [a(\vec{k}) + a^{\dagger}(-\vec{k})] \]

\[
\pi(\vec{x}) = -\frac{i}{L^d} \sum_{\vec{k} \in \Gamma} e^{i\vec{k} \cdot \vec{x}} \sqrt{\frac{\omega(\vec{k})}{2}} [a(\vec{k}) - a^{\dagger}(-\vec{k})] \]

Therefore,

\[
H_0 = \sum_{\vec{k} \in \Gamma} \mathcal{H}(\vec{k}), \quad \mathcal{H}(\vec{k}) = \frac{1}{L^d} \omega(\vec{k}) a^{\dagger}(\vec{k}) a(\vec{k})
\]

- All terms commute with each other.
- Each represents a harmonic oscillator.
Eigenvalue problem solved

\[ H|\{n(\vec{k}), \vec{k} \in \Gamma\}\rangle = E_{\{n(\vec{k}), \vec{k} \in \Gamma\}}|\{n(\vec{k}), \vec{k} \in \Gamma\}\rangle \]

where

\[ E_{\{n(\vec{k}), \vec{k} \in \Gamma\}} = \sum_{\vec{k} \in \Gamma} n(\vec{k}) \omega(\vec{k}) \]

Physical interpretation: \( n(\vec{k}) \) particles with momentum \( \vec{k} \)
(totals of \( \sum_{\vec{k} \in \Gamma} n(\vec{k}) \) particles).
Ground state has no particles and zero energy,

\[ H_0|0\rangle = 0 \]

One-particle states

\[ |\vec{k}\rangle \equiv \frac{1}{L^{d/2}} a^\dagger(\vec{k})|0\rangle \]

have energy

\[ H_0|\vec{k}\rangle = \omega(\vec{k})|\vec{k}\rangle \]
Lowest energy level: \( \omega(\vec{0}) = m \) (mass gap!) with corresponding normalized eigenstate

\[
|\vec{k} = 0\rangle = \frac{a^d}{L^{d/2}} \sum_{\vec{x} \in \Omega} \left[ \sqrt{\frac{m}{2}} \phi(\vec{x}) + \frac{i}{\sqrt{2m}} \pi(\vec{x}) \right] |0\rangle
\]

System can be simulated with a register of \( N \) qubits for each \( \vec{k} \in \Gamma \), if mapped onto basis states \( |n(\vec{k})\rangle \), \( n(\vec{k}) = 0, 1, \ldots, 2^N - 1 \).

Alternative simulation

Working in the \( \phi \) representation, we have

\[
\pi(\vec{x}) = -\frac{i}{a^d \partial \phi(\vec{x})} , \quad a(\vec{k}) = a^d \sum_{\vec{x} \in \Omega} e^{-i\vec{k} \cdot \vec{x}} \left[ \sqrt{\frac{\omega(\vec{k})}{2}} \phi(\vec{x}) + \frac{1}{\sqrt{2\omega(\vec{k})}} \frac{\partial}{\partial \phi(\vec{x})} \right]
\]

Define

\[
a(\vec{x}) = \frac{1}{L^d} \sum_{\vec{k} \in \Gamma} e^{i\vec{k} \cdot \vec{x}} \frac{1}{\sqrt{2\omega(\vec{k})}} a(\vec{k})
\]

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The ground state obeys
\[ a(\vec{x})\psi_0[\phi] = \frac{1}{2} \phi(\vec{x}) \psi[\phi] + \frac{1}{2\sqrt{-\nabla^2 + m^2}} \frac{\partial \psi[\phi]}{\partial \phi(\vec{x})} = 0 \, , \quad \forall x \in \Omega \]
therefore
\[ \psi_0[\phi] = C \exp \left\{ -\frac{a^d}{2} \sum_{\vec{x} \in \Omega} \phi(\vec{x}) \sqrt{-\nabla^2 + m^2} \phi(\vec{x}) \right\} \]
This Gaussian can be simulated by \( N \)-qubit registers at each site \( \vec{x} \in \Omega \). Other states can then be generated.
For one-particle states, introduce ancillary qubit and define
\[ H_1 = a^\dagger(\vec{k})|1\rangle\langle 0| + a(\vec{k})|0\rangle\langle 1| \]
We have
\[ H_1 \psi_0|0\rangle = \psi_1|1\rangle \, , \quad H_1 \psi_1|1\rangle = \psi_0|0\rangle \]
where \( \psi_1 = a^\dagger(\vec{k})\psi_0 \) is a one-particle state of momentum \( \vec{k} \), therefore
\[ e^{-iH_1\pi/2} \psi_0|0\rangle = -i\psi_1|1\rangle \]
**Simple interacting field theory**

Let us add an interaction with an external “charge” distribution $\rho(\vec{x})$, so

$$H = H_0 + H_\rho, \quad H_\rho = a^d \sum_{\vec{x} \in \Omega} \rho(\vec{x}) \phi(\vec{x})$$

In this case, the spectrum can be calculated exactly.

We have

$$H_\rho = \frac{1}{L^d} \sum_{\vec{k} \in \Gamma} \left[ \tilde{\rho}^*(\vec{k}) a(\vec{k}) + \tilde{\rho}(\vec{k}) a^\dagger(\vec{k}) \right]$$

where

$$\tilde{\rho}(\vec{k}) = a^d \sum_{\vec{x} \in \Omega} \frac{1}{\sqrt{2 \omega(\vec{k})}} \rho(\vec{x}) e^{i\vec{k} \cdot \vec{x}}$$

By defining new annihilation operator

$$b(\vec{k}) \equiv a(\vec{k}) + \frac{1}{\omega(\vec{k})} \tilde{\rho}(\vec{k})$$
we obtain

\[ H = \frac{1}{L^d} \sum_{\vec{k} \in \Gamma} \left[ \omega(\vec{k}) b^\dagger(\vec{k}) b(\vec{k}) - \frac{|\tilde{\rho}(\vec{k})|^2}{\omega(\vec{k})} \right] \]

The spectrum is as before, but with energy levels shifted by the new ground state energy

\[ E_0 = -\frac{1}{L^d} \sum_{\vec{k} \in \Gamma} \frac{|\tilde{\rho}(\vec{k})|^2}{\omega(\vec{k})} \]

In the limit \( m \to 0, a \to 0 \), this is simply the electrostatic energy of an electric charge distribution.

The eigenstates can be found from the eigenstates of the non-interacting system by switching on \( H_\rho \) adiabatically, or by acting with the new creation operators \( b^\dagger(\vec{k}) \) on the new ground state. The latter is easily shown to be the coherent state (not normalized)

\[ |0\rangle_\rho = \exp \left\{ -\frac{1}{L^d} \sum_{\vec{k} \in \Gamma} \frac{\tilde{\rho}(\vec{k})}{\omega(\vec{k})} a^\dagger(\vec{k}) \right\} |0\rangle \]
Quartic interaction

Hamiltonian

\[ H = a^d \sum_{\vec{x} \in \Omega} \left( \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} \phi(\vec{x})(-\nabla^2 + m_0^2)\phi(\vec{x}) + \frac{\lambda_0}{4!}\phi^4(\vec{x}) \right) \]

To prepare an initial state, start with the ground state of \( H_0 \), and adiabatically evolve it with \( H(t/\tau) \), where \( H(0) = H_0 \) and \( H(1) = H \).

One ends up with the ground state of \( H \), if \( \tau \) is long enough. The minimum \( \tau \) is determined by the mass gap \( m_0 \).

♣ What to do if \( m_0 = 0 \)?

The Path. Need to determine \( H(s) \) (\( 0 \leq s \leq 1 \)). Define

\[ H(s) = a^d \sum_{\vec{x} \in \Omega} \left( \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} \phi(\vec{x})(-\nabla^2 + m_0^2(s))\phi(\vec{x}) + \frac{\lambda_0(s)}{4!}\phi^4(\vec{x}) \right) \]

Simple choice: \( m_0^2(s) = m_0^2 \), \( \lambda_0(s) = s\lambda_0 \)

**Problem**: this choice does not span the entire physical parameter space.

- \( m_0^2 \) is not the physical mass. In fact, we can have \( m_0^2 < 0 \).

**Solution**: Use perturbation theory for an educated guess of the path.
Perturbation theory

Write $H = H_0 + H_I$, where unperturbed

$$H_0 = \frac{a^d}{2} \sum_{\vec{x} \in \Omega} \left( \pi^2(\vec{x}) + \phi(\vec{x})(-\nabla^2 + m^2)\phi(\vec{x}) \right)$$

($m^2$ is physical mass), and interaction Hamiltonian (perturbation)

$$H_I = a^d \sum_{\vec{x} \in \Omega} \left( \frac{\delta m^2}{2} \phi^2(\vec{x}) + \frac{\lambda_0}{4!} \phi^4(\vec{x}) \right)$$

mass counterterm: $\delta m = m_0^2 - m^2$.

A (somewhat involved) calculation of the mass gap yields

$$m^2 a^2 = \begin{cases} 
    m_0^2 a^2 + \frac{\lambda_0 a^2}{8\pi} \left( 1 - \frac{\lambda_0 a^2}{8\pi m_0^2 a^2} \right) \log \frac{64}{m_0^2 a^2} - \frac{(\lambda_0 a^2)^2}{384 m_0^2 a^2} + \ldots, & d = 1 \\
    m_0^2 a^2 + \frac{A_2}{16\pi^2} \left( 1 - \frac{\lambda_0 a}{16\pi m_0 a} \right) \lambda_0 a + \frac{\lambda_0^2 a^2}{48} \log m_0^2 a^2 + \ldots, & d = 2 \\
    m_0^2 a^2 + \frac{A_3}{32\pi^3} \left( 1 + \frac{\lambda_0}{32\pi^2} \log m_0^2 a^2 \right) \lambda_0 - \frac{B_3}{1536\pi^7} \lambda_0^2 + \ldots, & d = 3
\end{cases}$$
where (one-loop contribution)

\[ A_d = \int \cdots \int_{-\pi}^{\pi} \frac{d^d k}{2 \sqrt{\sum_{i=1}^{d} \sin^2 \frac{k_i}{2}}} \]

and (two-loop contribution)

\[ B_3 = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{\sqrt{\alpha(1-\alpha) + \beta(1-\alpha-\beta)}} \int \cdots \int_{-\pi}^{\pi} \frac{d^3 k d^3 k'}{\Delta^2} \]

\[ \Delta = 4 \sum_{i=1}^{d} \left[ \alpha \sin^2 \frac{k_i}{2} + \beta \sin^2 \frac{k'_i}{2} + (1-\alpha-\beta) \sin^2 \frac{k_i + k'_i}{2} \right] \]

Numerically, \( A_2 = 25.379 \ldots \), \( A_3 = 112.948 \ldots \).
Second-order phase transition at \( m^2 = 0, \lambda_0 = \lambda_c(m_0^2) \).

- Symmetric phase \((\phi \to -\phi)\) above curve \( m^2 = 0 \) (in \((\lambda_0, m_0^2)\) plane).
- Symmetry breaking below curve \( m^2 = 0 \).
- Need to choose path entirely above phase transition.
  - Efficient choice for weak coupling \( \lambda_0 \), using 1st-order (one-loop) perturbative result \((0 \leq s \leq 1)\),

\[
\lambda_0(s) = s \lambda_0, \quad m_0^2(s) = \begin{cases} 
  m_0^2 + \frac{(1-s)\lambda_0}{8\pi} \log \frac{64}{m_1^2 a^2}, & d = 1 \\
  m_0^2 + \frac{A_2}{16\pi^2} \frac{(1-s)\lambda_0}{a}, & d = 2 \\
  m_0^2 + \frac{A_3}{32\pi^3} \frac{(1-s)\lambda_0}{a^2}, & d = 3 
\end{cases}
\]

where \( m_1^2 \) is the one-loop estimate of the physical mass in \( d = 1 \),

\[
m_1^2 = m_0^2 + \frac{\lambda_0}{8\pi} \log \frac{64}{m_1^2 a^2}
\]

Works even if \( m_0^2 < 0 \).
– For strong coupling $\lambda_0$, we always have $m_0^2 > 0$, so choose

$$\lambda_0(s) = s\lambda_0, \quad m_0^2(s) = m_0^2$$

Near phase transition,

$$m^2 \sim |\lambda_0 - \lambda_c|^{2\nu}, \quad d < 3$$

critical exponent $\nu$ universal.

Numerically: $\nu = 1 \ (d = 1), \ \nu = 0.63 \ldots \ (d = 2)$. 

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QED

Lagrangian

\[ L = \frac{a^d}{2} \sum_{\vec{x} \in \Omega} (\vec{E}^2 - \vec{B}^2) \]

where

\[ \vec{E} = -\vec{\nabla}A_0 - \partial_0 \vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A} \]

Four fields, but we know photon is transverse, therefore only two degrees of freedom are physical.

System has *gauge symmetry* (local)

\[ A_0 \rightarrow A_0 - \partial_0 \chi, \quad \vec{A} \rightarrow \vec{A} + \vec{\nabla} \chi \]

Fix gauge by imposing

\[ \partial_0 A_0 + \vec{\nabla} \cdot \vec{A} = 0 \]

Introduce arbitrary parameter \( \lambda > 0 \), and modify Lagrangian

\[ L_\lambda = \frac{a^d}{2} \sum_{\vec{x} \in \Omega} \left[ \vec{E}^2 - \vec{B}^2 - \lambda(\partial_0 A_0 + \vec{\nabla} \cdot \vec{A})^2 \right] \]
Conjugate momenta to fields \((A_0, \vec{A})\),

\[
\pi_0(\vec{x}) = \frac{1}{a^d} \frac{\partial L_\lambda}{\partial (\partial_0 A_0(\vec{x}))} = -\lambda(\partial_0 A_0 + \vec{\nabla} \cdot \vec{A}), \quad \pi_i(\vec{x}) = \frac{1}{a^d} \frac{\partial L_\lambda}{\partial (\partial_0 A_i(\vec{x}))} = -E_i
\]

Notice \(\pi_0 = 0\), if \(\lambda = 0\) (constrained system).

No physical results should depend on \(\lambda\).

- For simplicity, set \(\lambda = 1\) (Feynman gauge).

Commutation relations

\[
[A_0(\vec{x}), \pi_0(\vec{y})] = i \frac{a^d}{a^d} \delta_{\vec{x}, \vec{y}} , \quad [A_i(\vec{x}), \pi_j(\vec{y})] = i \frac{a^d}{a^d} \delta_{ij} \delta_{\vec{x}, \vec{y}}
\]

Hamiltonian

\[
H = a^d \sum_{\vec{x} \in \Omega} \left[ -\pi_0 \partial_0 A_0 + \vec{\pi} \cdot \partial_0 \vec{A} \right] - L_\lambda
\]

\[
= \frac{a^d}{2} \sum_{\vec{x} \in \Omega} \left[ -\left(\pi_0 + \vec{\nabla} \cdot \vec{A}\right)^2 + A_0 \nabla^2 A_0 + (\vec{\pi} - \vec{\nabla} A_0)^2 - \vec{A} \cdot \nabla^2 \vec{A} \right]
\]

Hints of trouble:

- wrong sign of kinetic term \((\pi_0 + \ldots)^2\), and in \(\pi_0 = -\partial_0 A_0 + \ldots\)
As in scalar case, attempt to define “annihilation” operator for $A_0$,

$$a_0(\vec{k}) = a^d \sum_{\vec{x} \in \Omega} e^{-i\vec{k} \cdot \vec{x}} \left[ \sqrt{\frac{\omega(\vec{k})}{2}} A_0(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}} \left( \pi_0(\vec{x}) + \vec{\nabla} \cdot \vec{A}(\vec{x}) \right) \right]$$

where $\omega(\vec{k}) = \frac{2}{a} \sqrt{\sum_{i=1}^{d} \sin^2 \frac{k_i a}{2}}$. We have

$$H a_0^\dagger(\vec{k}) |0\rangle = -\omega(\vec{k}) a_0(\vec{k}) |0\rangle$$

Negative energy!

∴ We must define $a_0(\vec{k})$ to be the creation operator!

$$|\vec{k}, 0\rangle = \frac{1}{L^{d/2}} a_0(\vec{k}) |0\rangle$$

But then $\langle \vec{k}, 0 | \vec{k}, 0 \rangle = -1$. Negative norm state!

Need to restrict to transverse polarizations and reject unphysical states by imposing gauge fixing condition $\partial_0 A_0 + \vec{\nabla} \cdot \vec{A} = 0$.

We shall do this on the average:

$$\langle \Psi | \pi_0 | \Psi \rangle = 0$$
Introduce annihilation operators for $\vec{A}$,

$$\vec{a}(\vec{k}) = a^d \sum_{\vec{x} \in \Omega} e^{-i\vec{k} \cdot \vec{x}} \left[ \sqrt{\frac{\omega(\vec{k})}{2}} \vec{A}(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}} (\vec{\pi}(\vec{x}) - \vec{\nabla} A_0(\vec{x})) \right]$$

Hamiltonian (normal-ordered)

$$H = \sum_{\vec{k} \in \Gamma} H(\vec{k}) , \quad H(\vec{k}) = \frac{1}{L^d} \omega(\vec{k}) \left[ \vec{a}^\dagger(\vec{k}) \cdot \vec{a}(\vec{k}) - a_0(\vec{k}) a_0^\dagger(\vec{k}) \right]$$

Consider the general one-particle state (not normalized)

$$|\Psi\rangle = (\zeta_0 a_0(\vec{k}) + \zeta \cdot \vec{a}^\dagger(\vec{k})) |0\rangle$$

We have

$$\langle \Psi | \pi_0(\vec{x}) | \Psi \rangle \propto \omega(\vec{k}) \zeta_0 - \frac{2}{a} \sum_{i=1}^{d} \sin \frac{k_i a}{2} \zeta_i = 0$$

$\Rightarrow \zeta_0$ constrained.
Positive norm!

\[ \langle \psi | \psi \rangle \sim -|\zeta_0|^2 + |\zeta|^2 = |\zeta|^2 - \frac{|\kappa \cdot \zeta|^2}{\kappa^2} \geq 0 \]

where \( \kappa_i = \frac{2a}{a} \sin \frac{k_i a}{2} \).

Zero norm when \( \zeta = \bar{k}, \zeta_0 = \omega(k) \).

- Longitudinal polarization.
- Redundancy: if \( |\psi_0\rangle \) has zero norm, then \( |\psi\rangle \) and \( |\psi\rangle + |\psi_0\rangle \) describe the same physical system (due to gauge invariance).
- 2 physical degrees of freedom!

Addition of fixed charge (current) is handled as in the case of a scalar field.
OUTLOOK

- Quantum computation of high energy scattering amplitudes is faster than any classical algorithm (lattice field theory).
- What is the computational power of our Universe (QFTs)?
- Wilson discovered deep insights (renormalization group) in QFTs thinking about simulations on classical computers.
  - What insights will we gain with QC?
- Can QFC (quantum field computation) go beyond QC?
- Can we understand quantum gravity better with QC, or by thinking about information loss into a black hole?