Example of Final Test

(Each problem is 25 points)

① Given the Lagrange function \( L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 + \lambda \dot{q} q \) (a) find the Hamilton function; (b) Find the generator of the canonical transformation that convert this Hamilton function into the form \( P^2/2m + \frac{1}{2} kQ^2 \) of harmonic oscillator. Use Poisson brackets to verify explicitly that the transformation is canonical.

② Solve the Hamilton-Jacobi differential equation for a particle moving in a uniform gravitational field \( \ddot{g} \). From this solution find the system of algebraic equations \( \partial S/\partial \alpha_i = \beta_i \) and resolve coordinates as functions of time.

③ Show that Hamiltonian

\[
H(p,q,t) = \begin{cases} 
\frac{V(q)}{\gamma}, & 0 < t < \gamma T \\
\frac{p^2}{2(1-\gamma)}, & \gamma T < t < T 
\end{cases}
\]

, where \( T \) is a period of motion, results in an area-preserving map in the plane \((q,p)\)

④ Relativistic \( \pi^0 \) \((m_{\pi^0} = 135 \text{ MeV})\) meson with energy \( E_0 = 20 \text{ GeV} \) in the Lab decays in flight into two \( \gamma \)-rays: \( \pi^0 \rightarrow \gamma + \gamma \). (a) Find the energy of \( \gamma \)-ray \( E_\gamma \) as function of \( E_0 \) and the angle \( \theta \) (\( \theta \) is the Lab angle of corresponding \( \gamma \) with respect to the original direction of \( \pi^0 \)); (b) Find the minimum \( \theta_{\text{min}} \) and maximum \( \theta_{\text{max}} \) angle between two photons in the Lab \( (\theta = \theta_1 + \theta_2) \); (c) What are the lab energies of two final photons corresponding to \( \theta_{\text{min}} \) and \( \theta_{\text{max}} \) configurations?

⑤ An excited atom, of total mass \( m \), is at rest in a given frame. It emits a photon and thereby loses internal (i.e. rest) energy \( \Delta E \). Calculate the exact frequency of the photon, making due allowance for the recoil of the atom.